Midterm Exam 1.5 Solutions, Problem 1

First, graph vertex labels for vertices not on face.

Each graph vertex is a network link.

Connections to processors:

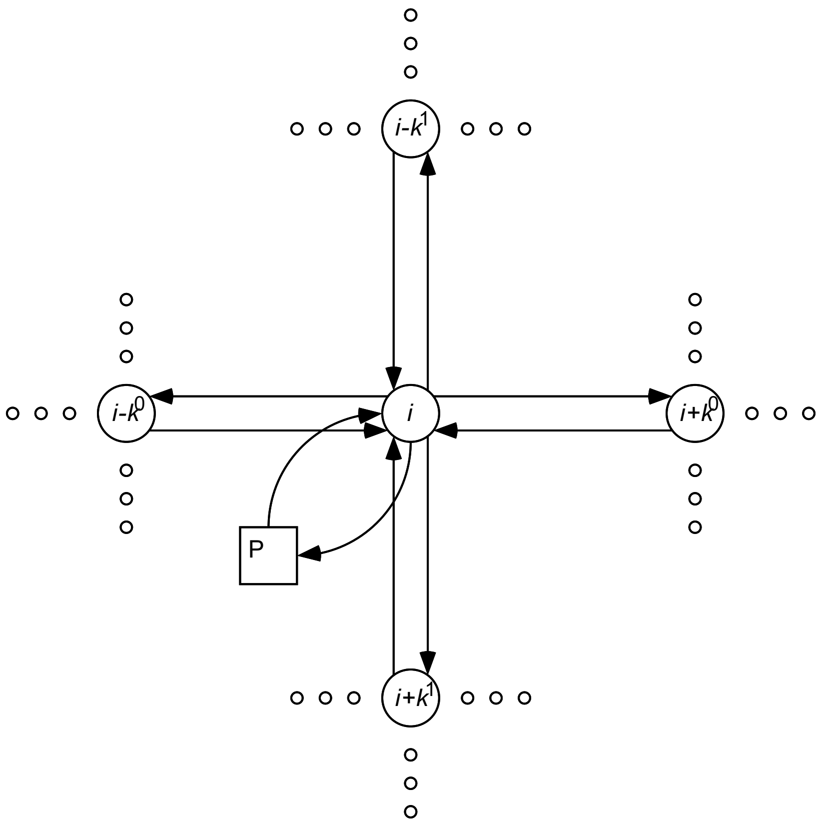
$$I = \{\, \langle i,\, \mathrm{I} \rangle \mid i \in \langle k^n \rangle \,\} \qquad O = \{\, \langle i,\, \mathrm{O} \rangle \mid i \in \langle k^n \rangle \,\}$$

Network-link (graph-vertex) label includes number of node link is leaving, $i \in \langle k^n \rangle$,

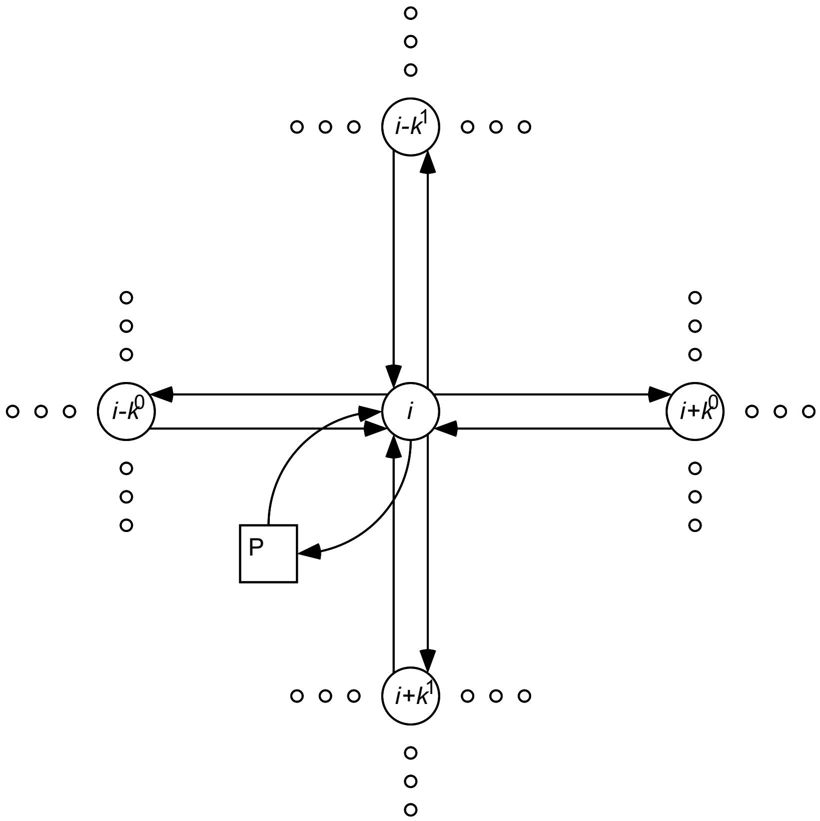
dimension along which link lies, $j \in \langle n \rangle$,

and direction $d \in \{-1, +1\}$:

$$V = I \cup O \cup \{\, \langle i,\, j,\, d \rangle \mid i \in \langle k^n \rangle \,,\ j \in \langle n \rangle \,,\ d \in \{-1, +1\} \,\}$$



The Edges



$$E = \{\, (\langle i,\, \mathrm{I}\rangle\,, \langle i,\, j,\, d\rangle) \mid i \in \langle k^n \rangle\,,\ j \in \langle n \rangle\,,\ d \in \{-1, +1\}\,\} \cup$$

$$\{\, (\langle i,\, j,\, d\rangle\,, \langle i',\, j',\, d'\rangle) \mid i \in \langle k^n \rangle\,,\ j \in \langle n \rangle\,,\ d \in \{-1, +1\},$$

$$i'_{(x)} = i_{(x)},\ x \neq j,\ i'_{(j)} = i_{(j)} + d,\ j' \in \langle n \rangle - \{j\},\ d' \in \{-1, +1\}\,\} \cup$$

$$\{\, (\langle i,\, j,\, d\rangle\,, \langle i',\, j,\, d\rangle) \mid j \in \langle n \rangle\,,\ d \in \{-1, +1\},$$

$$i \in \langle k^n \rangle\,,\ i'_{(x)} = i_{(x)},\ x \neq j,\ i'_{(j)} = i_{(j)} + d\,\} \cup$$

$$\{\, (\langle i,\, j,\, d\rangle\,, \langle i',\, \mathrm{O}\rangle) \mid i \in \langle k^n \rangle\,,\ j \in \langle n \rangle\,,\ d \in \{-1, +1\},$$

$$i'_{(x)} = i_{(x)},\ x \neq j,\ i'_{(j)} = i_{(j)} + d\ \}$$

Direction Set for All Vertices

The expression $d \in \{-1,+1\}$ is not appropriate for the faces of the KNC.

To handle all of the vertices, use the function OK $\mid \langle k^n\rangle \times \langle n\rangle \to \{\,\{-1,+1\},\{+1\},\{-1\}\,\}$:

$$\mathrm{OK}(i,j) = \begin{cases} \{-1,+1\}, & \text{if } i_{(j)} \neq 0 \text{ and } i_{(j)} \neq k-1; \\ \{+1\}, & \text{if } i_{(j)} = 0; \\ \{-1\}, & \text{if } i_{(j)} = k-1. \end{cases}$$

The Edge Set Using OK

$$E = \{\, (\langle i,\, \mathrm{I}\rangle\,, \langle i,\, j,\, d\rangle) \mid i \in \langle k^n\rangle\,,\ j \in \langle n\rangle\,,\ d \in \mathrm{OK}(i,j)\,\} \cup$$

$$\{\, (\langle i,\, j,\, d\rangle\,, \langle i',\, j',\, d'\rangle) \mid i \in \langle k^n\rangle\,,\ j \in \langle n\rangle\,,\ d \in \mathrm{OK}(i,j),$$

$$i'_{(x)} = i_{(x)},\ x \neq j,\ i'_{(j)} = i_{(j)} + d,\ j' \in \langle n\rangle - \{j\},\ d' \in \mathrm{OK}(i',j')\,\} \cup$$

$$\{\, (\langle i,\, j,\, d\rangle\,, \langle i',\, j,\, d\rangle) \mid i \in \langle k^n\rangle\,,\ j \in \langle n\rangle\,,\ d \in \mathrm{OK}(i,j),$$

$$i'_{(x)} = i_{(x)},\ x \neq j,\ i'_{(j)} = i_{(j)} + d\,\} \cup$$

$$\{\, (\langle i,\, j,\, d\rangle\,, \langle i',\, \mathrm{O}\rangle) \mid i \in \langle k^n\rangle\,,\ j \in \langle n\rangle\,,\ d \in \mathrm{OK}(i,j),$$

$$i'_{(x)} = i_{(x)},\ x \neq j,\ i'_{(j)} = i_{(j)} + d\ \}$$

Problem 2:

Since no messages are dropped, flow rate can be found by adding appropriate network-input arrival rates.

Consider the cell a request to the hot output, (a, H), passes through in stage x:

$$a_{(n-2-x)}a_{(n-3-x)}\ldots a_0 H_{(n-1)}H_{(n-2)}\ldots H_{(n-x)}a_{(n-x-1)}$$

Cell i in stage x will be in the saturation tree if:

$$i_{(x-1:0)} = H_{(n-1:n-x)}.$$

Number of inputs that can reach any link in this cell: $= m^x$.

If link (or cell) is part of saturation tree then it will receive hot messages from *all* connected inputs.

Because non-hot destinations are uniformly distributed, if the cell is not in the saturation tree the flow rate is:

$$\lambda(1-h)$$

If the cell is in the saturation tree:

$$\lambda(1-h) + \lambda h m^x$$

Problem 3

Part a:

Every stage has m^n links. The fewer stages, the fewer links.

Therefore, the minimum-cost omega network under this (admittedly unrealistic) model is a single crossbar.

Part b:

Two things must simultaneously occur:

- The queue must be full. (Prob. $= p_d$.)
- A message must arrive. (Prob. $= \lambda$.)

Probability that arrival dropped is then: $p_d\lambda$.

This can be expressed in terms of λ, μ, and d using the queue-state formulas.

Part c:

$$U = \min\left\{1, \frac{\frac{P}{N} t_{\mathrm{m}}}{t_{\mathrm{m}} + t_{\mathrm{nw}}}\right\} \tag{1}$$

What do we solve for?

"Rate at which processors send messages…"

According to the model, messages are sent after each task completes computation.

Computation takes time t_{m}.

Since the utilization is one, there is always a task computing.

Ergo, messages are sent at rate $1/t_{\mathrm{m}}$.

To get answer solve (1) for $1/t_{\mathrm{m}}$:

$$\text{Rate} \leq \frac{\frac{P}{N} - 1}{\sqrt{N}}$$