Solution to Problem 2

First, the correctly re-stated second part of the problem: Prove that if a shortest path is described by $L_1, R_1, L_2, 0, 0, \ldots$ in which $L_1 > 0$ and $L_2 > 0$ then $L_1 + L_2 \leq R_1$ and if a shortest path is described by $0, R_1, L_2, R_2, 0, 0, \ldots$ in which $R_1 > 0$ and $R_2 > 0$ then $R_1 + R_2 \leq L_2$.

An equivalent statement will be proven: if a path is described by $L_1, R_1, L_2, 0, 0, \ldots$ in which $L_1 > 0$ and $L_2 > 0$ and $L_1 + L_2 > R_1$ then there exists a shorter path connecting the two vertices. (A similar equivalent statement exists for the $0, R_1, L_2, R_2, 0, 0, \ldots$ case.)

There is an easy case, $R_1 \geq n$, and a tedious case, $R_1 < n$. In both cases, first find the vertex reached by path L_1, R_1, L_2, then find a shorter path reaching the same vertex.

Easy case: Replace L_1, R_1, L_2 with $n, 0, \ldots$ (n left shifts).

The tedious case is easy to argue graphically. First, understand that applying a path to a ring changes digits in the ring and possibly leaves the ring in a new position (*e.g.*, turned 90 deg). Any two paths that change the same digits and leave the ring in the same position are equivalent. A path of the form L_1, R_1, L_2 in which $L_1 + L_2 > R_1$ will "pass over" some segment of the ring three times. An alternate—and shorter—path can be found that starts in the opposite direction, R_1', L_2', R_2' and in which the segment that had been passed over three times is now passed over once; this can easily be seen graphically. This graphical argument is vague and so a more rigorous proof is needed.

In preparation for the proof, find the vertex reached using L consecutive left links and R consecutive right links. From the definition of the de-Bruijn network L consecutive shifts:

$$s_{(n-1)}s_{(n-2)}\cdots s_{(1)}s_{(0)} \rightarrow s_{(n-1-L)}s_{(n-2-L)}\cdots s_{(0)}x_{(n-1)}x_{(n-2)}\cdots x_{(n-L)}$$

where the starting vertex, s, and the ending vertex are written in radix m. The x refer to the links used (the digit exchanged in). Note that the subscripts refer to the numbering used in the starting node.

It will be useful to consider starting nodes of the form $s_{(a-1)}s_{(a-2)}\cdots s_{(0)}s_{(n-1)}s_{(n-2)}\cdots s_{(a)}$. Let $0 \leq L < n$ and $L \leq a < n$, then if L left shifts are used:

$$s_{(a-1)}s_{(a-2)}\cdots s_{(0)}s_{(n-1)}s_{(n-2)}\cdots s_{(a)} \rightarrow$$
$$s_{(a-1-L)}s_{(a-2-L)}\cdots s_{(0)}s_{(n-1)}s_{(n-2)}\cdots s_{(a)}x_{(a-1)}\cdots x_{(a-L)}.$$

For R right links:

$$s_{(n-1)}s_{(n-2)}\cdots s_{(1)}s_{(0)} \rightarrow x_{(R-1)}x_{(R-2)}\cdots x_{(0)}s_{(n-1)}s_{(n-2)}\cdots s_{(R)},$$

where the starting vertex, s, and the ending vertex are written in radix m. The x refer to the links used (the digit exchanged in). Note that the subscripts refer to the numbering used in the starting node.

It will also be useful for the right-link case to consider starting nodes of the form

$$s_{(a-1)}s_{(a-2)}\cdots s_{(0)}s_{(n-1)}s_{(n-2)}\cdots s_{(a)}.$$

Let $0 \leq R < n$ and $R \leq a < n$, then if R right shifts are used:

$$s_{(a-1)}s_{(a-2)}\cdots s_{(0)}s_{(n-1)}s_{(n-2)}\cdots s_{(a)} \rightarrow x_{(a-1+R)}x_{(a-2+R)}\cdots x_{(a)}s_{(a-1)}\cdots s_{(0)}s_{(n-1)}\cdots s_{(a+R)}.$$

Using these, the vertex reached using path L_1, R_1, L_2 (starting at s) will be found.

The path will be divided into five segments by splitting R_1 and L_2, each into two pieces to form path $L_1, R_{11}, 0, R_{12}, L_{21}, 0, L_{22}$. Note that this is exactly the same path. The starting vertex and the verticies reached after each segment are:
start

$$s_{(n-1)}s_{(n-2)}\cdots s_{(0)}$$

left L_1

$$s_{(n-1-L_1)}s_{(n-2-L_1)}\cdots s_{(0)}x_{(n-1)}x_{(n-2)}\cdots x_{(n-L_1)}$$

right $R_{11} = L_1$

$$x_{(n-1)}x_{(n-2)}\cdots x_{(n-L_1)}s_{(n-1-L_1)}s_{(n-2-L_1)}\cdots s_{(1)}s_{(0)}$$

right $R_{12} = R_1 - L_1$

$$x_{(R_1-L_1-1)}x_{(R_1-L_1-2)}\cdots x_{(0)}x_{(n-1)}\cdots x_{(n-L_1)}s_{(n-L_1-1)}\cdots s_{(R_1-L_1)}$$

left $L_{21} = R_1 - L_1$

$$x_{(n-1)}x_{(n-2)}\cdots x_{(n-L_1)}s_{(n-1-L_1)}\cdots s_{(R_1-L_1)}x_{(R_1-L_1-1)}\cdots x_{(0)}$$

left $L_{22} = L_1 + L_2 - R_1$

$$x_{(n-1-L_1-L_2+R_1)}x_{(n-2-L_1-L_2+R_1)}\cdots x_{(n-L_1)}s_{(n-1-L_1)}\cdots$$
$$s_{(R_1-L_1)}x_{(R_1-L_1-1)}\cdots x_{(0)}x_{(n-1)}\cdots x_{(n-L_1-L_2+R_1)}$$

Now, an alternate path, $0, R'_1, L'_2, R'_2$ will be given reaching the same node. Let $R'_1 = R_1 - L_1$, $L'_2 = R_1$, and $R'_2 = R_1 - L_2$. Now, because $L_1 + L_2 > R_1$ the original path length, $L_1 + R_1 + L_2$, is longer than $2R_1$. The alternate path is of length $3R_1 - L_1 - L_2 < 2R_1$, and so shorter than the original. Using this path, with L'_1 and R'_2 split to illustrate parts of the path:
start

$$s_{(n-1)}s_{(n-2)}\cdots s_{(0)}$$

right $R'_1 = R_1 - L_1$

$$x_{(R_1-L_1-1)}x_{(R_1-L_1-2)}\cdots x_{(0)}s_{(n-1)}\cdots s_{(R_1-L_1)}$$

left $L'_{11} = R'_1$

$$s_{(n-1)}s_{(n-2)}\cdots s_{(R_1-L_1)}x_{(R_1-L_1-1)}\cdots x_{(0)}$$

left $L'_{12} = L_1$

$$s_{(n-1-L_1)}s_{(n-2-L_1)}\cdots s_{(R_1-L_1)}x_{(R_1-L_1-1)}\cdots x_{(0)}x_{(n-1)}\cdots x_{(n-L_1)}$$

right $R'_2 = R_1 - L_2$

$$x_{(n-1-L_1-L_2+R_1)}x_{(n-2-L_1-L_2+R_1)}\cdots x_{(n-L_1)}s_{(n-1-L_1)}\cdots$$
$$s_{(R_1-L_1)}x_{(R_1-L_1-1)}\cdots x_{(0)}x_{(n-1)}\cdots x_{(n-L_1-L_2+R_1)}$$

which is the same destination. (End of proof.)

Using this result we can prove that a shortest path must be in the form $L_1, R_1, L_2, 0, 0, \ldots$ or $0, R_1, L_2, R_2, 0, 0, \ldots$. First it will be shown that any path can be converted into a form in which all non-zero elements are consecutive. In other words there are no zeros between non-zero L's and R's. Consider a path of the form $\ldots, R_i, 0, 0, L_{i+2}, \ldots$. Clearly the two zeros can be removed to form an identical path $\ldots R_i, L_{i+2}, \ldots$, with all elements starting at L_{i+2} shifting two places to the left. Now consider $\ldots, R_i, 0, R_{i+1}, \ldots$. This is equivalent to $\ldots, R_i + R_{i+1}, \ldots$, with all elements starting at R_{i+1} shifted two places to the left. (Only R_{i+1} is added to another element.) Applying

these rules to any finite sequence will yield a path in which all non-zero elements are consecutive and in which the distance is the same as the original path.

Now it will be shown that any path consisting of consecutive non-zero elements can be replaced by one of the form $L_1, R_1, L_2, 0, 0, \ldots$ or $0, R_1, L_2, R_2, 0, 0, \ldots$ which is no longer than the original.

Consider sequence $L_1, R_1, L_2, R_2, \ldots$ in which the symbols shown all have positive values. It's easy to show that one of the two inequalities must be true: $L_1 + L_2 > R_1$ or $R_1 + R_2 > L_2$. If the first holds then using the result proven above, the sequence can be replaced by the shorter sequence $0, R_1', L_2', R_2' + R_2, \ldots$, which also specifies a shorter path. A similar substitution can be used when the second inequality holds. Since these substitutions can be applied to any list of four or more elements a shortest path need only consist of three or fewer consecutive elements.(End of proof.)