EE 7725 **Homework 6** **Solution 1996**

Solution 1: Consider $(a,b) \in P$ and $(b,c) \in Q$. Since $R = PQ$, $(a,c) \in R$. Because $P \in \Upsilon$ there exist f_x such that

$$b_{(n-x-1)} = a_{(n-x-1)} f_x(b_{(n-1)} \cdots b_{(n-x)}). \tag{1}$$

Because $Q \in \Upsilon$ there exist g_x such that

$$c_{(n-x-1)} = b_{(n-x-1)} g_x(c_{(n-1)} \cdots c_{(n-x)}). \tag{2}$$

To prove $R \in \Upsilon$ must prove the existence of h_x such that

$$c_{(n-x-1)} = a_{(n-x-1)} h_x(c_{(n-1)} \cdots c_{(n-x)}). \tag{3}$$

Replacing $b_{(n-x-1)}$ in (2) with the right-hand side of (1) yields:

$$c_{(n-x-1)} = a_{(n-x-1)} f_x(b_{(n-1)} \cdots b_{(n-x)}) g_x(c_{(n-1)} \cdots c_{(n-x)}). \tag{4}$$

Equating the functions in (3) and (4):

$$h_x(c_{(n-1)} \cdots c_{(n-x)}) = f_x(b_{(n-1)} \cdots b_{(n-x)}) g_x(c_{(n-1)} \cdots c_{(n-x)}). \tag{5}$$

We're not done because h_x is supposed to map $c_{(n-1)} \cdots c_{(n-x)}$ while the RHS of (5) also uses $b_{(n-1)} \cdots b_{(n-x)}$. A function is needed that maps $c_{(n-1)} \cdots c_{(n-x)}$ to $b_{(n-1)} \cdots b_{(n-x)}$. This mapping can be found from (2). For example, when $x = 0$, $c_{(n-1)} = b_{(n-1)} g_0()$ and so $b_{(n-1)} = c_{(n-1)} [g_0()]^{-1}$. Remember that $[g_0()]^{-1}$ is the inverse of the value, not the function. When $x = 1$, $c_{(n-2)} = b_{(n-2)} g_1(c_{(n-1)})$ and so $b_{(n-2)} = c_{(n-2)} \left[g_1(c_{(n-1)})\right]^{-1}$, etc. Thus the b's can be found in terms of c's using g, which is known to exist.

Call the function mapping $c_{(n-1)} \cdots c_{(n-x)}$ to $b_{(n-1)} \cdots b_{(n-x)}$, g'_x. Then

$$h_x(c_{(n-1)} \cdots c_{(n-x)}) = f_x(g'_x(c_{(n-1)} \cdots c_{(n-x)}) g_x(c_{(n-1)} \cdots c_{(n-x)});$$

since h_x can be found R is in Υ.

Solution 2: A permutation P of $\langle m^n \rangle$ is a *baseline-network upper-triangular permutation* if there exist functions $f_x \mid \langle m^x \rangle \rightarrow \Sigma_m$ such that for all $(a,\alpha) \in P$

$$\alpha_{(n-x-1)} = a_{(x)} f_x(\alpha_{(n-1)} \cdots \alpha_{(n-x)}).$$

A permutation P of $\langle m^n \rangle$ is a *baseline-network lower-triangular permutation* if there exist functions $f_x \mid \langle m^x \rangle \rightarrow \Sigma_m$ such that for all $(a,\alpha) \in P$

$$\alpha_{(n-x-1)} = a_{(x)} f_x(a_{(n-1)} \cdots a_{(x+1)}).$$

Solution 3: Let $P \in \Upsilon_{m,n,B}$. Then there exist f_x such that for all $(a, \alpha) \in P$

$$\alpha_{(n-x-1)} = a_{(x)} f_x(\alpha_{(n-1)} \cdots \alpha_{(n-x)}).$$

To show $P^{-1} \notin \Upsilon$ prove that no functions g_x exist such that

$$a_{(n-x-1)} = \alpha_{(x)} g_x(a_{(n-1)} \cdots a_{(n-x)}).$$

Consider $x = n - 1$. For $(a, \alpha) \in P$:

$$\alpha_{(0)} = a_{(n-1)} f_{n-1}(\alpha_{(n-1)} \cdots \alpha_{(1)})$$

Consider $(\alpha, a) \in P^{-1}$ and $x = 0$:

$$a_{(n-1)} = \alpha_{(0)} g_0().$$

In P^{-1} the most-significant digit of the source, $a_{(n-1)}$, is a permutation of the least-significant digit of the destination, $\alpha_{(0)}$. In P it's possible to choose f_{n-1} so that $\alpha_{(0)}$ depends upon not only $a_{(n-1)}$ but also $\alpha_{(n-1)} \cdots \alpha_{(1)}$. No g_0 could have $\alpha_{(0)}$ depend upon anything other than $a_{(n-1)}$, and so g_0 does not exist for all permutations, therefore $P \in \Upsilon \not\Leftrightarrow P^{-1} \in \Upsilon$.