Sets-of-Permutations Representation

Network is described by its permutation connection assignments.

E.g., a crossbar can realize all permutations.

Networks can also be defined in terms of permutation sets.

Mathematical properties of permutations well known.

This knowledge can then be applied to network.

Definitions

Symbols represent network and cell inputs and outputs.

Symbols will be written as lower-case Roman letters or digits.

For example: a, b, z, 0, 9.

Permutations represent link patterns, crossbar states, and permutation connection assignments.

Permutations are written as upper case Roman or lower case Greek letters.

For example: A, Z, α, ω.

Mathematically, a permutation is a one-to-one mapping from a set of symbols on to itself.

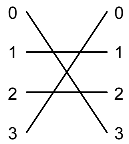
Double Row Notation for Permutations

First row contains symbols, second row contains image of symbols.

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & 2 & 0 \end{pmatrix}$$

In this example, 0's image is 3.

The permutation above represents the following:



In general,

$$P = \begin{pmatrix} 0 & 1 & \ldots & N-1 \\ d_0 & d_1 & \ldots & d_{N-1} \end{pmatrix},$$

where $d_i \in \langle N \rangle$ for $i \in \langle N \rangle$ and $d_i \neq d_j$ if $i \neq j$.

Composition

Common questions:

Where does link pattern P connect stage terminal x?

What is the result of cascading link patterns P and Q?

These questions are expressed mathematically using the *composition* operator:

xP for first question, PQ for second.

Composition can be applied to symbols, permutations, and sets of permutations.

Composition of Symbols with Permutations

Definition: The *composition* of a symbol a with a permutation P:

is written aP.

$aP = b$ if P *maps* a to b.

That is, $P = \begin{pmatrix} 0 & 1 & \ldots & a & \ldots \\ d_0 & d_1 & \ldots & b & \ldots \end{pmatrix}$.

Composition of Permutations with Permutations

Definition: The composition of permutation P with permutation Q:

is written PQ;

the result of the operation is a permutation R such that:

$$\text{for all } a \in \langle N \rangle \quad (aP)Q = aR$$

Example

Let $P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 0 \end{pmatrix}$.

Then $PQ = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 1 & 3 \end{pmatrix}$ and $QP = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$.

Composition is not commutative.

Composition is associative.

Familiar Permutations

The *identity permutation* of $\langle N \rangle$:

$$I_N = \begin{pmatrix} 0 & 1 & \dots & N-1 \\ 0 & 1 & \dots & N-1 \end{pmatrix}$$

The m, k *shuffle permutation* of $\langle mk \rangle$:
for all symbols $a \in \langle N \rangle$, $a\sigma_{m,k} = b$ if $b \equiv (ma + \lfloor a/k \rfloor) \bmod mk$.

All functions that were used for link patterns are permutations.

Inverse Permutation

Inverse Permutation

Let P be a permutation.

The *inverse* of permutation P, denoted P^{-1}, ...

... is the permutation for which $PP^{-1} = I$ is true ...

... where I is the identity permutation.

In words: if P and P^{-1} are represented as link patterns ...

... then P^{-1} would be the mirror image of P.

Example:

If $P = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 3 & 2 & 0 \end{pmatrix}$ then $P^{-1} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 2 & 1 \end{pmatrix}$.

Properties

- Every permutation has exactly one inverse.
- $PP^{-1} = P^{-1}P = I$.

Sets of Permutation

Consider $\Sigma_2 = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

This set contains two permutations...

...in fact, those realizable by a 2×2 crossbar.

Such sets used to represent networks and cells.

Usually denoted using upper case Greek or upper-case calligraphic Roman letters.

$\mathcal{F}\Omega\mathcal{R}\ \mathcal{EXAM}\Pi\Lambda\mathcal{E}$

Each permutation represents a connection assignment that network or cell can realize.

The Symmetric Group

Commonly needed: set of all permutations.

Called the *symmetric group*.

Denoted Σ_N, for permutations over $\langle N \rangle$.

For example:

$$\Sigma_3 = \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \right\}$$

Denoted Σ_A, for permutations over symbols in set A.

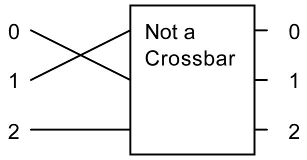
For example:

$$\Sigma_{\{a,b\}} = \left\{ \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \begin{pmatrix} a & b \\ b & a \end{pmatrix} \right\}.$$

Note, $\Sigma_N = \Sigma_{\langle N \rangle}$.

Composition of Permutation and Set

Consider:



Let P denote the permutation representing the link pattern.

Let $\mathcal{A}$ denote the sets of permutation realizable by the cell.

Then $P = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}$ and $\mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \right\}$.

Network's permutations obtained by composing P and $\mathcal{A}$ is denoted $P\mathcal{A}$.

The result of composing permutation P

with set of permutation $\mathcal{A}$ is defined to be

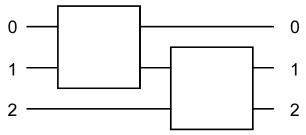
$$P\mathcal{A} = \{\, PA \mid A \in \mathcal{A} \}.$$

For the Example

$$P\mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \right\}.$$

Composition of Two Sets of Permutations

Consider:



$\Sigma_2 = \mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$.

$\Sigma_{\{1,2\}} = \mathcal{B} = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$.

(Note that omitted symbols are mapped to themselves.)

The composition of two permutation sets is defined:

$\{AB | A \in \mathcal{A},\ B \in \mathcal{B}\}$.

Example:

Consider $\mathcal{AB}$.

$\mathcal{A} = \left\{ \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$

$\mathcal{B} = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$

$\mathcal{AB} = \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \right\}$

Composition of a Set with Itself

Let $\mathcal{A}$ be a set of permutations.

Let i be a positive integer.

Then the notation $\mathcal{A}^i$ is defined to be

$$\prod_{j=1}^{i} \mathcal{A}$$

For example,

$\mathcal{A}^3 = \mathcal{AAA}$.

Representation of 2×2, n-stage Omega Network

Goal: Obtain set of permutations for:

2^n-input omega network using 2×2 cells.

Will call the set, $\Omega_{2,n}$.

Cells:

Permutations realizable by cell j in stage i:

$\Sigma_{\{2j,2j+1\}}$.

Cells in a Stage:

Will call permutations realizable by cells in a stage $\mathcal{E}_{2,n}$.

$$\mathcal{E}_{2,n} = \prod_{j=0}^{2^{n-1}-1} \Sigma_{\{2j,2j+1\}}.$$

This omits shuffle.

Entire Stage:

Permutations by any stage: $\left(\sigma_{2,2^{n-1}} \mathcal{E}_{2,n}\right)$.

Entire Network:

$\Omega_{2,n} = \left(\sigma_{2,2^{n-1}} \mathcal{E}_{2,n}\right)^n$.