Direct Network Graph Representation

Idea: describe (specify) direct network using graph.

Network $\Leftrightarrow$ Graph

Links $\Leftrightarrow$ Edges

Nodes $\Leftrightarrow$ Vertices

Graph Representation

Uses two sets:

- Set of vertices, V.
- Set of edges E.

Usual notation, $G = (V, E)$:

- G is the name of the graph.
- V is the set of vertices.
- E is the set of edges.

Let $u \in V$ and $v \in V$.

The following two statements are equivalent:

- There is an edge between u and v.
- $(u, v) \in E$.

For any graph (V, E): $E \subseteq V \times V$.

Notation

For graphs used in class:

V is a set of consecutive integers starting at 0.

E.g., $V = \{0, 1, 2\}$.

Integers sometimes used in radix-k form.

Let i be an integer.

Let k be an integer and $a \in \langle k \rangle$.

Then notation $i_{(a)}$ indicates digit a in i's radix-k representation.

Digit 0 is the least significant, digit $k-1$ is the most significant.

Other Useful Notation

$\langle x \rangle \equiv \{0, 1, \ldots, x-1\}$.

E.g., $\langle 4 \rangle \equiv \{0, 1, 2, 3\}$.

k-ary n-cube (KNC) Network Family

Popular direct network family.

Also called: n-dimensional mesh, generalized cube.

Properties

- All members are easily routed.
- Members exhibit medium to high diameter.
- Suitable for many parallel algorithms.

Plan

- Special Cases Described
- Family Described

Linear Network

Nodes arranged in a line.

Graph description of N-node linear network:

$$V = \langle N \rangle = \{0, 1, \ldots, N-1\}$$

$$E = \{\, (i, i+1) \mid i \in \langle N-1 \rangle \,\}$$

Properties

Degree, $\delta = 2$.

Routing: increment (or decrement) vertex of current position until destination reached.

Distance, $d_{u,v} = |u - v|$.

Diameter, $D = N - 1$.

Average Distance:

$$\overline{d} = \frac{2}{N(N-1)} \sum_{i=0}^{N-2} \sum_{j=i+1}^{N-1} j - i = \frac{N+1}{3}.$$

Bisection width, 1.

Usefulness

Diameter and average distance too large for general use.

Bisection width too small for general use.

Might be useful for special-purpose applications.

Useful for simple classroom examples.

Two-Dimensional Mesh Network

Generalization of linear network to two dimensions.

Graph description of k^2-node 2-dimensional mesh:

$$V = \langle k^2 \rangle = \{0, 1, \ldots, k^2 - 1\}$$

$$\begin{aligned} E =& \{\, (i, i+1) \mid i \in \langle k^2 \rangle,\ (i \bmod k) < k - 1 \,\} \\ & \cup \{\, (i, i+k) \mid i \in \langle k^2 - k \rangle \,\} \end{aligned}$$

Properties

Degree, $\delta = 4$.

Routing:

- Treat vertex as 2-digit, radix-k number.
- Increment (or decrement) least-significant digit of vertex of current position until equal to least-significant digit of destination vertex.
- Increment (or decrement) most-significant digit of vertex of current position until equal to most-significant digit of destination vertex.

Distance, $d_{u,v} = \left|u_{(0)} - v_{(0)}\right| + \left|u_{(1)} - v_{(1)}\right|$.

Diameter, $D = 2(k-1)$.

Average Distance:

$$\overline{d} = \frac{2\,(-1+k)\,k\,(1+k)}{3\,(-1+k^2)}.$$

Bisection width, k.

Usefulness

Diameter large, but acceptable.

May be easy to build.

Used in general-purpose computers.

Well suited to some algorithms.

Works poorly with other algorithms.

KNC: Generalization of previous two networks to n dimensions.

Graph description of k^n-node k-ary n-cube:

$V = \langle k^n \rangle$.

$E = \{\, (u, u + k^i) \mid i \in \langle n \rangle\,,\; u \in V,\; u_{(i)} < k - 1 \,\}$.

Properties

Degree:

$$\delta = \begin{cases} 2n, & \text{if } k > 2; \\ n, & \text{if } k = 2. \end{cases}$$

Routing:

- Treat vertex as n-digit, radix-k number.
- Choose a digit.
- Increment (or decrement) this digit of the vertex of the current position until equal to the corresponding digit of destination vertex.
- Repeat until all digits are chosen.

Distance, $d_{u,v} = \sum_{i=0}^{n-1} \left| u_{(i)} - v_{(i)} \right|$.

Diameter, $D = n(k-1)$.

Average Distance:

$$\overline{d} = \tfrac{n}{3} \tfrac{(k-1)(k+1)k^{n-1}}{(k^n-1)} \approx \tfrac{n}{3} \left(k - \tfrac{1}{k} \right).$$

Bisection width, k^{n-1}.

Usefulness For $k = 2$.

Short (logarithmic) average distance and diameter.

Easy to route.

Large degree (bad).

Choice of k and n for KNC

Goal: determine tradeoffs when k and n varied.

Method:

- *Fix* some measures.
- *Vary* k and n.
- *Observe* effect on other measures.

Case 1: Fix N.

Observe effect on average distance, latency, bisection width, and cost.

Case 2: Fix N and Bisection Width

Observe effect on average distance, latency, and cost.

Case 3: Fix N and Cost

Observe effect on average distance, latency, and bisection width.

Equations for All Cases

Cost (For These Comparisons).

Count number of links, include width:

$$\mathrm{C} = \frac{1}{2} N \delta w.$$

Number of Nodes

$N = k^n$ by definition of KNC.

$\Rightarrow n = \log_k N$

$\Rightarrow k = N^{\frac{1}{n}}$

Latency (For These Comparisons)

$$L = \overline{d} + \frac{M}{w} - 1$$

Moore Bound

Motivation: does the hypercube have a minimum diameter?

Can use Moore bound to answer this.

Method Outline:

- Fix degree and diameter of minimum-diameter network.

  (Degree of δ, diameter of d).

- Find maximum number of nodes that any such network could have.

- Solve for diameter.

Derivation

Call some node the *center* of the network.

Let $N'(i)$ denote the number of nodes at distance i from center.

Then:

$$N'(0) = 1$$

$$N'(1) \leq \delta$$

$$N'(2) \leq \delta(\delta - 1) = N'(1)(\delta - 1)$$

$$N'(3) \leq \delta(\delta - 1)^2 = N'(2)(\delta - 1)$$

$$N'(i) \leq N'(i-1)(\delta - 1) = \delta(\delta - 1)^{(i-1)} \text{ for } i > 1.$$

Let N denote the total number of nodes in a network of diameter d and degree δ.

$$
\begin{aligned}
N &= \sum_{i=0}^{d} N'(i) \\
&\le N'(0) + N'(1) + \sum_{i=2}^{d} N'(i) \\
&\le 1 + \delta + \delta \sum_{i=2}^{d} (\delta - 1)^{i-1} \\
&\le 1 + \delta + \delta \sum_{j=1}^{d-1} (\delta - 1)^{j} \\
&\le 1 + \delta + \delta \left(\frac{(\delta-1)^d - (\delta - 1)}{(\delta - 1) - 1} \right) \\
&\le \frac{\delta(\delta-1)^d - 2}{\delta - 2}
\end{aligned}
$$

Solving for d yields:

$$d \ge \log_{(\delta-2)} \left(\frac{N(\delta-2)+2}{\delta} \right).$$

If $\delta \gg 1$:

$$d \approx \log_\delta N.$$

Note that this is much better than the KNC family.

But do such networks exist?

Shuffle and Shift Functions

Used to describe edges in several graphs.

Idea: Rotate digits in a number (with an end-around shift).

Two definitions will be given:

Shuffle Function (for Integers)

Let $u \in \langle mk \rangle$ where m and k are positive integers.

The shuffle function $\sigma_{m,k} \mid \langle mk \rangle \to \langle mk \rangle$ is given by:

$$\sigma_{m,k}(u) \equiv mu + \left\lfloor \frac{u}{k} \right\rfloor \pmod{mk}.$$

Examples:

$$\sigma_{2,4}(1) = 2$$

$$\sigma_{2,4}(0) = 0$$

$$\sigma_{2,4}(5) = 3$$

Shift Functions (for Sequences)

Let $u_{(n-1)}\, u_{(n-2)}\ \ldots\ u_{(0)}$ be any sequence of symbols, where $u_{(i)} \in S$, and $\mathcal{S}$ be the set of all possible sequences.

Then the left-shift function $\sigma_\mathrm{l} \mid \mathcal{S} \to \mathcal{S}$ is given by:

$$\sigma_\mathrm{l}\left(u_{(n-1)}\, u_{(n-2)}\ \ldots\ u_{(0)}\right) = u_{(n-2)}\, u_{(n-3)}\ \ldots\ u_{(0)}\, u_{(n-1)}.$$

Examples:

$$\sigma_\mathrm{l}(abc) = bca$$

$$\sigma_\mathrm{l}(1101) = 1011$$

The right-shift function $\sigma_\mathrm{r} \mid \mathcal{S} \to \mathcal{S}$ is given by:

$$\sigma_\mathrm{r}\left(u_{(n-1)}\, u_{(n-2)}\ \ldots\ u_{(0)}\right) = u_{(0)}\, u_{(n-1)}\ \ldots\ u_{(2)}\, u_{(1)}.$$

Examples:

$$\sigma_\mathrm{r}(abc) = cab$$

$$\sigma_\mathrm{r}(1101) = 1110$$

Relationship Between Shuffle and Shift Functions

The shuffle function is a special case of the shift functions.

That is:

- Given any set of symbols S,
- any positive integer n,
- and any set of sequences $\mathcal{S} = S \times S \times \cdots \times S$ (n times),
- and any $S \in \mathcal{S}$,

there exist a corresponding:

- set of digits $\langle |S| \rangle$,
- set of integers $\langle |S|^n \rangle$,
- a mapping $\mathcal{S} \to \langle |S|^n \rangle$

such that for any $S_1 \in \mathcal{S}$

if $\sigma_{\mathrm{l}}(S_1) = S_2$ and $\sigma_{\mathrm{r}}(S_1) = S_3$

then $\sigma_{|S|,|S|^{n-1}}(s_1) = s_2$ and $\sigma_{|S|^{n-1},|S|}(s_1) = s_3$

where s_1, s_2, and s_3 are the integers corresponding to S_1, S_2, and S_3, respectively.

In other words, any sequence of symbols could be viewed as a sequence of digits.

The Exchange Function

Used to describe edges in several graphs.

Idea: change least-significant-digit of a number.

Exchange function for Integers

Let $u \in \langle mk \rangle$ and $i \in \langle m \rangle$, where m and k are positive integers.

Then the exchange function $\chi \mid \langle mk \rangle, \langle m \rangle \to \langle mk \rangle$ is given by

$$\chi_m(u, i) = m \left\lfloor \frac{u}{m} \right\rfloor + i$$

Examples:

$$\chi_2(5, 0) = 4$$

$$\chi_3(13, 2) = 14$$

Exchange function for Sequences

Let $u_{(n-1)}\, u_{(n-2)}\, \ldots\, u_{(0)}$ be any sequence of symbols, where $u_{(i)} \in S$,

and $\mathcal{S}$ be the set of all possible sequences.

Then the exchange function $\chi \mid \mathcal{S}, S \to \mathcal{S}$ is given by:

$$\chi(u_{(n-1)}\, u_{(n-2)}\, \ldots\, u_{(0)}, x) = u_{(n-1)}\, u_{(n-2)}\, \ldots\, x, \text{ where } x \in S.$$

Examples:

$$\chi(abc, d) = abd$$

$$\chi(1011, 0) = 1010$$

$$\chi(\heartsuit\clubsuit\diamondsuit, \spadesuit) = \heartsuit\clubsuit\spadesuit$$

Shuffle-Exchange Graph

Let m and n be positive integers.

The m,n shuffle-exchange, (V,E) graph is given by:

$V = \langle m^n \rangle$

$E = \{\, (u, \sigma_{m,m^{n-1}}(u)) \mid u \in V \,\} \cup \{\, (u, \chi_m(u,i)) \mid u \in V,\ i \in \langle m \rangle \,\}.$

Characteristics:

Degree: $\delta = m + 1$.

Distance: $d_{u,v} \leq 2n - 1$.

The exact distance cannot be expressed in compact form.

Diameter: $D = 2n - 1$.

For example, $d_{0,m^n-1} = 2n - 1$.

Average distance: not known, probably close to diameter.

Bisection width: for $m = 2$: $BW = \Theta(2^{n-1})$.

Non-minimal Routing of Shuffle Exchange Graph

For request $(u_{(n-1)}u_{(n-2)}\cdots u_{(0)},\ v_{(n-1)}v_{(n-2)}\cdots v_{(0)})$:

- Step 0:

  "Replace" least-significant-digit of source with MSD of destination.

  $(u_{(n-1)}u_{(n-2)}\cdots u_{(0)},\ u_{(n-1)}u_{(n-2)}\cdots v_{(n-1)})$.

- Step $i \in \{1, 2, \ldots, n-1\}$:

  Left shift the current node number.

  $$\begin{aligned}&\big(u_{(n-i)}u_{(n-i-1)}\cdots u_{(1)}v_{(n-1)}\cdots v_{(n-i)},\\ &\quad u_{(n-i-1)}u_{(n-i-2)}\cdots u_{(1)}v_{(n-1)}\cdots u_{(n-i)}\big)\end{aligned}$$

  "Replace" digit $n-i$ of source with digit $n-i-1$ of destination.

  Take edge

  $$\begin{aligned}&\big(u_{(n-i-1)}u_{(n-i-2)}\cdots u_{(1)}v_{(n-1)}\cdots u_{(n-i)},\\ &\quad u_{(n-i-1)}u_{(n-i-2)}\cdots u_{(1)}v_{(n-1)}\cdots v_{(n-i-1)}\big)\end{aligned}$$

de Bruijn Graph

Also called Good graph.

Let m and n be positive integers.

The m, n de Bruijn Graph, (V, E) is given by:

$V = \langle m^n \rangle$

$E = \{\, (u, \chi_m(\sigma_{m,m^{n-1}}(u), i)) \mid u \in V \; i \in \langle m \rangle \,\}.$

Characteristics:

Degree: $\delta = 2m$.

Distance: $d_{u,v} \leq n$.

The exact distance cannot be expressed in a compact form.

Diameter: $D = n$.

For example, $d_{0,m^n-1} = n$.

Average distance:

$$\overline{d} \geq \begin{cases} n - 3 - \frac{9}{8}, & \text{if } m = 2; \\ n - 1 - \frac{8}{9}, & \text{if } m = 3; \\ n - 1 - \frac{25}{72}, & \text{if } m = 4; \\ n - \frac{2(m+1)^2}{m(m-1)^2}, & \text{if } m > 4. \end{cases}$$

Non-minimal Routing of the de Bruijn Graph

For request $(u_{(n-1)}u_{(n-2)}\cdots u_{(0)},\ v_{(n-1)}v_{(n-2)}\cdots v_{(0)})$:

- Step $i \in \{0,1,\ldots,n-1\}$:

  Left-shift the current node number, then exchange LSD.

$$\big(u_{(n-i-1)}u_{(n-i-2)}\cdots u_{(0)}v_{(n-1)}\cdots v_{(n-i)},\\ u_{(n-i-2)}u_{(n-i-3)}\cdots u_{(0)}v_{(n-1)}\cdots v_{(n-i-1)}\big)$$