Equivalence

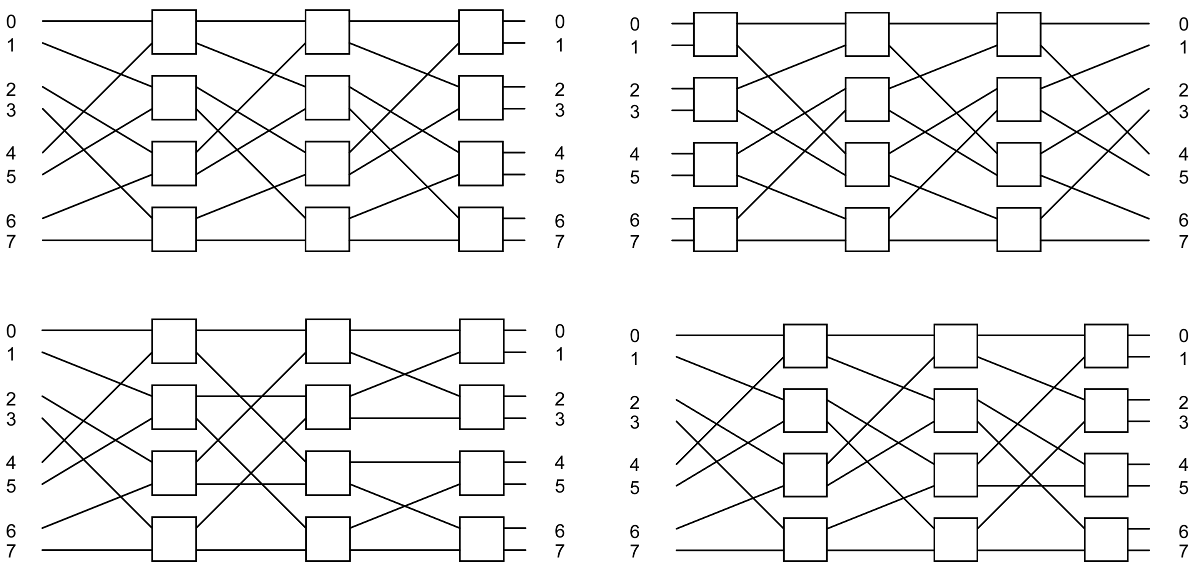
Two networks can look different but be the same.

They can also appear similar, but be different.

For example:

All of the networks below are banyans.

The networks have other similarities and differences. . .



Equivalence refers to sameness.

Defined in several different ways

Descriptively Equivalent (DE): drawn the same. (Link and cell numbers match.)

Without a doubt, identical.

Terminal Equivalent (TE): same cell and link connections when input and output numbers preserved but cells and links possibly renamed.

Identical, even though they may be drawn differently.

Topologically Equivalent (OE): same cell and link connections when input and output numbers ignored.

Can satisfy same connection assignments if inputs renamed.

Functionally Equivalent (FE): can satisfy same connections assignments.

Two networks do the same thing but have different topologies.

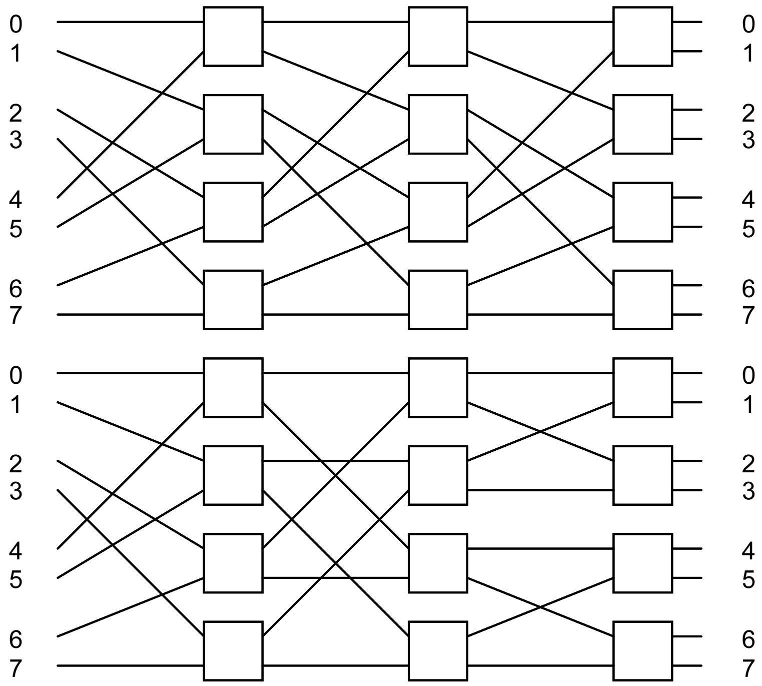
Descriptive Equivalence

Two networks are *descriptively equivalent (DE)* when there is a one-to-one correspondence between cells, links, and their labels, (including positions), when the networks are rendered in their usual way.

The relational operator $\stackrel{\mathrm{DE}}{=}$ indicates terminal equivalence, $A \stackrel{\mathrm{DE}}{=} B$ means A is descriptively equivalence to B.

Informally, this type of equivalence is a *no-brainer.*

The following two networks are *not* descriptively equivalent:

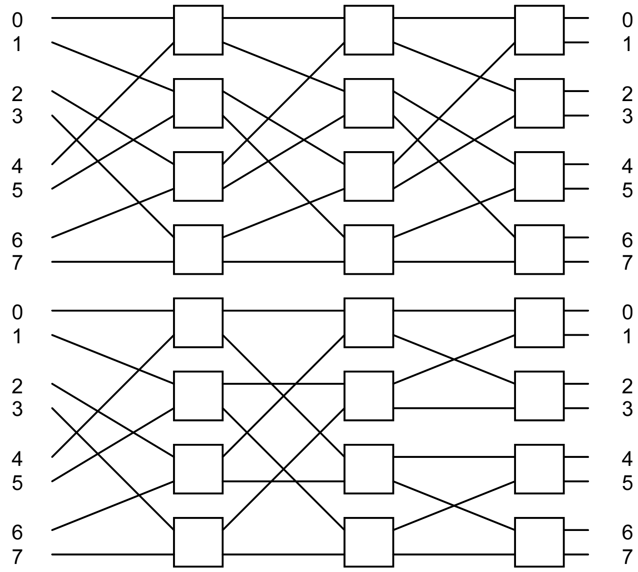


Terminal Equivalence

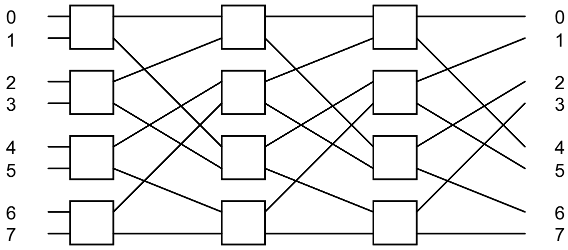
Two networks are *terminal equivalent (TE)* if their LGM representations are identical for some renaming of vertices in $V - (I \cup O)$.

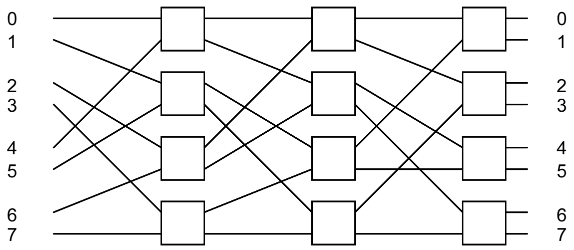
The relational operator $\stackrel{\mathrm{TE}}{=}$ indicates terminal equivalence; $A \stackrel{\mathrm{TE}}{=} B$ means A is terminal equivalent to B.

The following two networks are terminal equivalent:



Each of the two below is not TE to any of the others:





Determining if Two Networks are TE

Let $A = (I_A, O_A, V_A, E_A)$ and $B = (I_B, O_B, V_B, E_B)$ be network LGMs.

Easy cases: Two networks are *not* TE if:

- Either $|I_A| \neq |I_B|$, $|O_A| \neq |O_B|$, $|V_A| \neq |V_B|$, or $|E_A| \neq |E_B|$. (The number of inputs, outputs, cells, and links must be identical for TE.)

or

- Either $I_A \neq I_B$ or $O_A \neq O_B$. (Input and output labels must be identical for TE.)

Other cases will need a *mapping function*...

Tool needed to show equivalence: *mapping function*.

Maps one set of labels onto another.

Notation: $f \mid X \to Y$, symbols in X mapped to Y.

Example:

$$X = \{\text{One}, \text{Two}, \text{Three}\} \qquad Y = \{2, 1, 3\}$$

$$f(x) = \begin{cases} 1, & \text{if } x = \text{One}; \\ 2, & \text{if } x = \text{Two}; \\ 3, & \text{if } x = \text{Three}. \end{cases}$$

If $f(x) = y$ then y is called x's *image* under f;

In the function above, 1 is the image of One.

Mapping function must be *bijective*, *id est*, each element of X must be mapped to exactly one element of Y and vice versa.

The function above is bijective.

The following mapping function is **not** bijective:

$$X = \{\text{mouth}, \text{hand}, \text{foot}\} \qquad Y = \{\text{paw}, \text{snout}, \text{tail}\}$$

$$f(x) = \begin{cases} \text{paw}, & \text{if } x = \text{hand}; \\ \text{paw}, & \text{if } x = \text{foot}; \\ \text{snout}, & \text{if } x = \text{mouth}. \end{cases}$$

Numerical example: $Y = X = \langle mk \rangle$,

$$f(x) = \sigma_{m,k}(x).$$

Notation for Mapping Function

Let $X = \{\text{One}, \text{Two}, \text{Three}\}$ and $Y = \{2, 1, 3\}$

$$f(x) = \begin{cases} 1, & \text{if } x = \text{One}; \\ 2, & \text{if } x = \text{Two}; \\ 3, & \text{if } x = \text{Three}. \end{cases}$$

Usual notation: $f(\text{One}) = 1$

Mapping can be applied to sets:

$$f(A) = \{\, f(a) \mid a \in A \,\}.$$

For example: $f(X) = Y$ and

$$\begin{aligned} f(\{\text{One}, \text{Two}, \text{Three}\}) &= \{f(\text{One}), f(\text{Two}), f(\text{Three})\} \\ &= \{1, 2, 3\}. \end{aligned}$$

Mapping can be applied to tuples:

$$f(\,(A, B, C)\,) = (f(A), f(B), f(C)).$$

Tuples can be network LGMs, (I, O, V, E), edges, (a, b), or anything else.

To Prove That Two Networks Are TE Using Mapping Functions

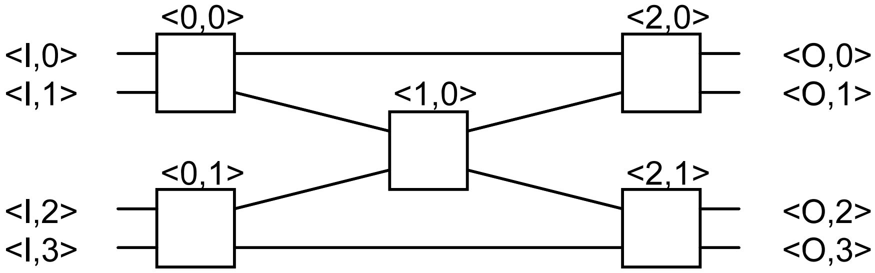
Let $A = (I_A, O_A, V_A, E_A)$ and $B = (I_B, O_B, V_B, E_B)$ be network LGMs.

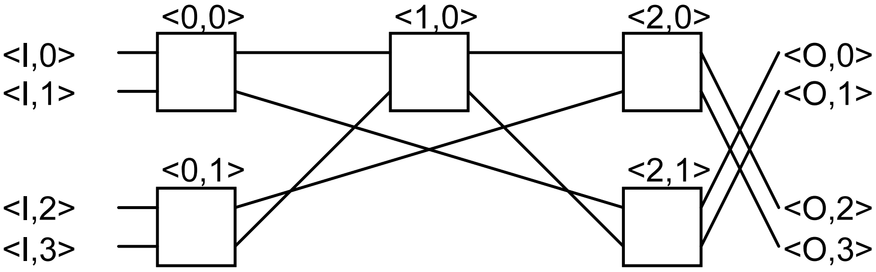
Network $A \stackrel{\text{TE}}{=} B$ iff

$I_A = I_B$ and $O_A = O_B$

and there exists a bijection $f \mid V_A \to V_B$ such that $f(A) = B$.

Simple Example: Prove that these two networks are identical:





Input and output labels are same for both networks (or they wouldn't be TE).

$$I_A = I_B = \{\langle \mathrm{I},\, 0\rangle, \langle \mathrm{I},\, 1\rangle, \langle \mathrm{I},\, 2\rangle, \langle \mathrm{I},\, 3\rangle\}$$
$$O_A = O_B = \{\langle \mathrm{O},\, 0\rangle, \langle \mathrm{O},\, 1\rangle, \langle \mathrm{O},\, 2\rangle, \langle \mathrm{O},\, 3\rangle\}$$

Edge labels are different, of course:

$$\begin{aligned} E_A = \{&(\langle \mathrm{I},\, 0\rangle, \langle 0,\, 0\rangle)\,(\langle \mathrm{I},\, 1\rangle, \langle 0,\, 0\rangle)\,(\langle \mathrm{I},\, 2\rangle, \langle 0,\, 1\rangle)\,(\langle \mathrm{I},\, 3\rangle, \langle 0,\, 1\rangle),\\ &(\langle 0,\, 0\rangle, \langle 2,\, 0\rangle),\ (\langle 0,\, 0\rangle, \langle 1,\, 0\rangle),\ (\langle 0,\, 1\rangle, \langle 1,\, 0\rangle),\ (\langle 0,\, 1\rangle, \langle 2,\, 1\rangle),\\ &(\langle 1,\, 0\rangle, \langle 2,\, 0\rangle),\ (\langle 1,\, 0\rangle, \langle 2,\, 1\rangle),\\ &(\langle 2,\, 0\rangle, \langle \mathrm{O},\, 0\rangle),\ (\langle 2,\, 0\rangle, \langle \mathrm{O},\, 1\rangle),\ (\langle 2,\, 1\rangle, \langle \mathrm{O},\, 2\rangle),\ (\langle 2,\, 1\rangle, \langle \mathrm{O},\, 3\rangle)\}\\ E_B = \{&(\langle \mathrm{I},\, 0\rangle, \langle 0,\, 0\rangle),\ (\langle \mathrm{I},\, 1\rangle, \langle 0,\, 0\rangle),\ (\langle \mathrm{I},\, 2\rangle, \langle 0,\, 1\rangle),\ (\langle \mathrm{I},\, 3\rangle, \langle 0,\, 1\rangle),\\ &(\langle 0,\, 0\rangle, \langle 1,\, 0\rangle),\ (\langle 0,\, 0\rangle, \langle 2,\, 1\rangle),\ (\langle 0,\, 1\rangle, \langle 1,\, 0\rangle),\ (\langle 0,\, 1\rangle, \langle 2,\, 0\rangle),\\ &(\langle 1,\, 0\rangle, \langle 2,\, 0\rangle),\ (\langle 1,\, 0\rangle, \langle 2,\, 1\rangle),\\ &(\langle 2,\, 0\rangle, \langle \mathrm{O},\, 2\rangle),\ (\langle 2,\, 0\rangle, \langle \mathrm{O},\, 3\rangle),\ (\langle 2,\, 1\rangle, \langle \mathrm{O},\, 0\rangle),\ (\langle 2,\, 1\rangle, \langle \mathrm{O},\, 1\rangle)\} \end{aligned}$$

Mapping Function for this Simple Example

The "easy" part:

Because $(\langle \mathrm{I},\, 0\rangle, \langle 0,\, 0\rangle) \in E_A$, and $(\langle \mathrm{I},\, 0\rangle, \langle 0,\, 0\rangle) \in E_B$:

$$f(\langle 0,\, 0\rangle) = \langle 0,\, 0\rangle$$

For similar reasons

$$f(\langle 0,\, 1\rangle) = \langle 0,\, 1\rangle$$

Because $(\langle 2,\, 0\rangle, \langle \mathrm{O},\, 0\rangle) \in E_A$, and $(\langle 2,\, 1\rangle, \langle \mathrm{O},\, 0\rangle) \in E_B$:

$$f(\langle 2,\, 0\rangle) = \langle 2,\, 1\rangle$$

For similar reasons

$$f(\langle 2,\, 1\rangle) = \langle 2,\, 0\rangle$$

As always for TE $f(\langle \mathrm{I},\, i\rangle) = \langle \mathrm{I},\, i\rangle$ and $f(\langle \mathrm{O},\, i\rangle) = \langle \mathrm{O},\, i\rangle$.

The part that would be hard (if the network were more complicated).

We need to find a mapping for the one remaining cell.

Since the map must be bijective, we have only one choice:

$$f(\langle 1,\, 0\rangle) = \langle 1,\, 0\rangle$$

Now that we have a mapping function, we can find $f(A)$

Use of Mapping Function in the Simple Example

By definition,

$$f(A) = (I_A, O_A, f(V_A), f(E_A)) \stackrel{?}{=} B$$

That is, to show $A \stackrel{\text{TE}}{=} B$ must show that $f(A) = B$.

Clearly, $f(V_A) = V_B$.

Check $f(E_A)$:

$$\begin{aligned}
f(E_A) = \{\, & (f(\langle \text{I}, 0\rangle), f(\langle 0, 0\rangle)),\ (f(\langle \text{I}, 1\rangle), f(\langle 0, 0\rangle)), \\
& (f(\langle \text{I}, 2\rangle), f(\langle 0, 1\rangle)),\ (f(\langle \text{I}, 3\rangle), f(\langle 0, 1\rangle)), \\
& (f(\langle 0, 0\rangle), f(\langle 2, 0\rangle)),\ (f(\langle 0, 0\rangle), f(\langle 1, 0\rangle)), \\
& (f(\langle 0, 1\rangle), f(\langle 1, 0\rangle)),\ (f(\langle 0, 1\rangle), f(\langle 2, 1\rangle)), \\
& (f(\langle 1, 0\rangle), f(\langle 2, 0\rangle)),\ (f(\langle 1, 0\rangle), f(\langle 2, 1\rangle)), \\
& (f(\langle 2, 0\rangle), f(\langle \text{O}, 0\rangle)),\ (f(\langle 2, 0\rangle), f(\langle \text{O}, 1\rangle)), \\
& (f(\langle 2, 1\rangle), f(\langle \text{O}, 2\rangle)),\ (f(\langle 2, 1\rangle), f(\langle \text{O}, 3\rangle))\,\} \\
= \{\, & (\langle \text{I}, 0\rangle, \langle 0, 0\rangle),\ (\langle \text{I}, 1\rangle, \langle 0, 0\rangle), \\
& (\langle \text{I}, 2\rangle, \langle 0, 1\rangle),\ (\langle \text{I}, 3\rangle, \langle 0, 1\rangle), \\
& (\langle 0, 0\rangle, \langle 2, 1\rangle),\ (\langle 0, 0\rangle, \langle 1, 0\rangle), \\
& (\langle 0, 1\rangle, \langle 1, 0\rangle),\ (\langle 0, 1\rangle, \langle 2, 0\rangle), \\
& (\langle 1, 0\rangle, \langle 2, 1\rangle),\ (\langle 1, 0\rangle, \langle 2, 0\rangle), \\
& (\langle 2, 1\rangle, \langle \text{O}, 0\rangle),\ (\langle 2, 1\rangle, \langle \text{O}, 1\rangle), \\
& (\langle 2, 0\rangle, \langle \text{O}, 2\rangle),\ (\langle 2, 0\rangle, \langle \text{O}, 3\rangle)\,\}
\end{aligned}$$

Therefore, the two networks are TE.

 6-12

Exhaustive Procedure for Determining if Two Networks are TE:

Let $A = (I_A, O_A, V_A, E_A)$ and $B = (I_B, O_B, V_B, E_B)$ be network LGMs.

1: Return "not TE" if either $I_A \neq I_B$, $O_A \neq O_B$, $|V_A| \neq |V_B|$, or $|E_A| \neq |E_B|$.

2: Otherwise, let $C_A = V_A - I_A - O_A$ and $C_B = V_B - I_B - O_B$, the cell labels.

There are $|C_A|!$ bijective mapping functions $f \mid C_A \rightarrow C_B$.

Let the functions be numbered and let f_i denote function i, for $0 \leq i < |C_A|!$.

3: Set $i = 0$.

4: **Loop:** Create LGM A' by applying f_i to each cell label of A.

5: If $A' = B$ then return "A and B are TE."

6: Set $i = i + 1$.

7: If $i < |C_A|!$ then goto **Loop**

8: Return "LGM A and B are not TE."

 EE 7725 Lecture Transparency. Formatted 14:01, 15 October 1996 from lsli12. 6-12

Efficient Procedures for Determining if Two Networks are TE:

Fact which will be exploited:

$\boxed{\textit{Mapping of first- and last-stage cells is unique.}}$

Let $A = (I_A, O_A, V_A, E_A)$ and $B = (I_B, O_B, V_B, E_B)$ be network LGMs.

Consider only networks in which each input and output is connected to exactly one cell.

If $A \stackrel{\mathrm{TE}}{=} B$ then:

For all $\langle \mathrm{I},\, i\rangle \in I_A$,

and $(\langle \mathrm{I},\, i\rangle, \langle x,\, j\rangle) \in E_A$,

and $(\langle \mathrm{I},\, i\rangle, \langle x',\, j'\rangle) \in E_B$:

$$f(\langle x,\, j\rangle) = \langle x',\, j'\rangle \tag{1}$$

In words: For TE networks A and B, if input i connects to cell $\langle x,\, j\rangle$ in network A, and input i connects to cell $\langle x',\, j'\rangle$ in network B, then cell $\langle x,\, j\rangle$ maps to cell $\langle x',\, j'\rangle$.

Similarly, if $A \stackrel{\mathrm{TE}}{=} B$ then:

For all $\langle \mathrm{O},\, i\rangle \in O_A$,

and $(\langle x,\, j\rangle\,, \langle \mathrm{O},\, i\rangle) \in E_A$,

and $(\langle x',\, j'\rangle\,, \langle \mathrm{O},\, i\rangle) \in E_B$:

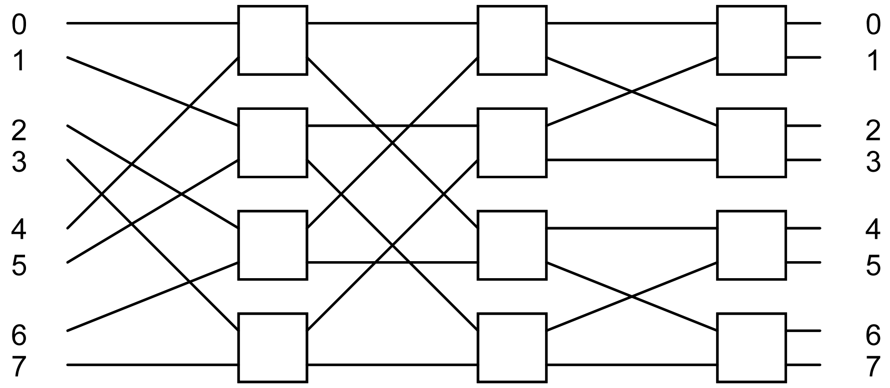
$$f(\langle x,\, j\rangle) = \langle x',\, j'\rangle \tag{2}$$

In words: For TE networks A and B, if output i connects to cell $\langle x,\, j\rangle$ in network A, and output i connects to cell $\langle x',\, j'\rangle$ in network B, then cell $\langle x,\, j\rangle$ maps to cell $\langle x',\, j'\rangle$.

Therefore, when choosing the f_i, select only those that satisfy equations (1,2)

Proving Terminal Equivalence of Omega and Recursive Omega

Introducing the recursive omega network:



Structure of an n-stage $m \times m$-cell recursive omega network:

- Each stage has m^{n-1} cells.
- Network inputs connect to stage-0 cells with an m, m^{n-1} shuffle.
- Stage-x cell outputs connect to stage-$(x+1)$ cell inputs with a $n-x-1$ *butterfly*, for $0 \leq x < n-1$.
- Stage-$(n-1)$ cell outputs connect to like-numbered network outputs.

The radix-m, n-digit, i butterfly function, $\beta_{(i)} \mid \langle m^n \rangle \to \langle m^n \rangle$ is given by

$$\beta_{(i)}(x) = \begin{cases} x_{(n-1:2)} x_{(0)} x_{(1)}, & \text{if } i = 1; \\ x_{(n-1:i+1)} x_{(0)} x_{(i-1:1)} x_{(i)}, & \text{if } 1 < i < n-1; \\ x_{(0)} x_{(n-2:1)} x_{(n-1)}, & \text{if } i = n-1; \end{cases}$$

for $0 < i < n$ and $0 \leq x < m^n$.

In words: digits 0 and i are swapped.

LGM of Recursive Omega (RΩ):

Let $k = m^{n-1}$, as usual.

$$I_{\mathrm{R}\Omega} = \{\, \langle \mathrm{I},\, i\rangle \mid 0 \le i < mk \,\} \qquad O_{\mathrm{R}\Omega} = \{\, \langle \mathrm{O},\, i\rangle \mid 0 \le i < mk \,\}$$

$$V_{\mathrm{R}\Omega} = I_{\mathrm{R}\Omega} \cup O_{\mathrm{R}\Omega} \cup \{\, \langle x,\, i\rangle \mid 0 \le x < n,\ 0 \le i < k \,\}$$

So far, just like an omega network.

The links set them apart:

$$\begin{aligned} E_{\mathrm{R}\Omega} = &\left\{ \left(\langle \mathrm{I},\, i\rangle, \langle 0,\, i \bmod k\rangle\right) \mid 0 \le i < mk \right\} \cup \\ &\left\{ \left(\langle x,\, i\rangle, \left\langle x+1,\, \left\lfloor \frac{\beta_{(n-x-1)}(m\,i+d)}{m} \right\rfloor \right\rangle\right) \right| \\ &\qquad\qquad 0 \le d < m,\ 0 \le i < k,\ 0 \le x < n-1 \Bigg\} \cup \\ &\{\, \left(\langle n-1,\, i\rangle, \langle \mathrm{O},\, mi+j\rangle\right) \mid 0 \le j < m,\ 0 \le i < k \,\} \end{aligned}$$

Non-input/output links can also be written as:

$$\begin{aligned} \{\, &\left(\langle x,\, i\rangle, \left\langle x+1,\, i_{(n-2)} i_{(n-3)} \ldots i_{(n-x-1)} d\, i_{(n-x-3)} \ldots i_{(0)} \right\rangle\right) \mid \\ &\qquad\qquad 0 \le d < m,\ 0 \le i < k,\ 1 \le x < n-1 \,\} \end{aligned}$$

LGM of an $(n, m, m^{n-1}, \mathrm{S}, \mathrm{T})$ Omega Network (For Review)

Let $k = m^{n-1}$.

$$I_\Omega = \{\, \langle \mathrm{I},\, i\rangle \mid 0 \le i < mk \,\} \qquad O_\Omega = \{\, \langle \mathrm{O},\, i\rangle \mid 0 \le i < mk \,\}$$

$$V_\Omega = I_\Omega \cup O_\Omega \cup \{\, \langle x,\, i\rangle \mid 0 \le x < n,\; 0 \le i < k \,\}$$

$$\begin{aligned} E_\Omega = &\{\, (\langle \mathrm{I},\, i\rangle, \langle 0,\, i \bmod k\rangle) \mid 0 \le i < mk \,\} \cup \\ &\{\, (\langle x,\, i\rangle, \langle x+1,\, mi+j \bmod k\rangle) \mid \\ &\quad 0 \le j < m,\; 0 \le i < k,\; 0 \le x < n-1 \,\} \cup \\ &\{\, (\langle n-1,\, i\rangle, \langle \mathrm{O},\, mi+j\rangle) \mid 0 \le j < m,\; 0 \le i < k \,\} \end{aligned}$$

Outline of Proof that Recursive Omega is TE to Omega

- Prove that the input and output labels are identical. (Easy)
- Find part of the mapping function for first- and last-stage cells. (Easy)

  (This part of the mapping function could be used to show that two networks are not TE.)
- Find the remainder of the mapping function. (Interesting)

Proof That Input and Output Labels Identical

By definition of the networks.

Mapping Function for First- and Last-Stage Cells

The first and last stages in the two networks are identical.

Therefore, $f(\langle 0, i\rangle) = \langle 0, i\rangle \ldots$

$\ldots$and $f(\langle n-1, i\rangle) = \langle n-1, i\rangle$ for $0 \le i < m^{n-1}$.

Mapping Function for Stage-1 Cells

Consider a link from stage-0 cell i to stage 1 for both networks:

In graph notation:

$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle, \left\langle 1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d \right\rangle\right) \in E_{\Omega}$$

$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle, \left\langle 1,\, d\, i_{(n-3)} \ldots i_{(0)} \right\rangle\right) \in E_{\mathrm{R}\Omega}$$

In "path" notation:

$$\begin{array}{cc} \Omega & \mathrm{R}\Omega \\ \left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle & \left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle \\ \left\langle 1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d \right\rangle & \left\langle 1,\, d\, i_{(n-3)} \ldots i_{(0)} \right\rangle \end{array}$$

for $0 \leq d < m$.

Note: The notation shows the links from stage-0 *cell* i. Similar notation has been used for showing the links taken by a request (*e.g.*, (a, α)). Regardless of the stage, cell i's radix-m representation is $i_{(n-2)} i_{(n-3)} \ldots i_{(0)}$.

Mapping function for these stage-1 cells:

$$\begin{aligned} f\left(\langle 1,\, i\rangle\right) = f\left(\left\langle 1,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle\right) &= \left\langle 1,\, i_{(0)} i_{(n-2)} \ldots i_{(1)} \right\rangle \\ &= \left\langle 1,\, \sigma_{m^{n-2},m}(i) \right\rangle \end{aligned}$$

Using the mapping function:

The mapping function (from the previous transparency):

$$f\left(\langle 1,\, i\rangle\right) = f\left(\left\langle 1,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle\right) = \left\langle 1,\, i_{(0)} i_{(n-2)} \ldots i_{(1)}\right\rangle$$
$$= \left\langle 1,\, \sigma_{m^{n-2},m}(i)\right\rangle$$

The links in the two networks (also from the previous transparency):

$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle, \left\langle 1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d\right\rangle\right) \in E_{\Omega}$$

$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle, \left\langle 1,\, d\, i_{(n-3)} \ldots i_{(0)}\right\rangle\right) \in E_{\mathrm{R}\Omega}$$

for $0 \le d < m$.

The mapping function used on Ω to (hopefully) obtain $\mathrm{R}\Omega$:

$$\left(f\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle\right), f\left(\left\langle 1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d\right\rangle\right)\right) \in f(E_{\Omega})$$
$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle, \left\langle 1,\, d\, i_{(n-3)} \ldots i_{(0)}\right\rangle\right) \in E_{\mathrm{R}\Omega}$$

Thus, we have chosen so that links between stage 0 and 1 in the two networks will match.

Mapping Function for Stage-2 Cells

Link from stage-1 cell i to stage 2 for the omega network:

$$\left(\left\langle 1,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle, \left\langle 2,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d \right\rangle\right) \in E_{\Omega}$$

for $0 \le d < m$.

In the recursive omega consider $f(\langle 1,\, i\rangle)$, cell i's image:

$$f\left(\left\langle 1,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle\right) = \left\langle 1,\, i_{(0)} i_{(n-2)} \ldots i_{(1)} \right\rangle$$

The link from this cell to stage 2 in the RΩ:

$$\left(\left\langle 1,\, i_{(0)} i_{(n-2)} i_{(n-3)} \ldots i_{(1)} \right\rangle, \left\langle 2,\, i_{(0)} d\, i_{(n-3)} \ldots i_{(1)} \right\rangle\right) \in E_{\mathrm{R}\Omega}$$

for $0 \le d < m$.

To show $\Omega \stackrel{\mathrm{TE}}{=} \mathrm{R}\Omega$ we must find maps between these stage-2 cells:

$$\begin{aligned} f(\langle 2,\, i\rangle) = f\left(\left\langle 2,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle\right) &= \left\langle 2,\, i_{(1)} i_{(0)} i_{(n-2)} \ldots i_{(2)} \right\rangle \\ &= \left\langle 2,\, \sigma_{m^{n-3},m^2}(i) \right\rangle \end{aligned}$$

Using this function:

$$\begin{aligned} \left(f(\left\langle 1,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle), f(\left\langle 2,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d \right\rangle)\right) &\in f(E_{\Omega}) \\ \left(\left\langle 1,\, i_{(0)} i_{(n-2)} i_{(n-3)} \ldots i_{(1)} \right\rangle, \left\langle 2,\, i_{(0)} d\, i_{(n-3)} \ldots i_{(1)} \right\rangle\right) &\in E_{\mathrm{R}\Omega} \end{aligned}$$

Mapping Function for Stage-$x+1$ Cells

Link from stage-x cell i to stage $x+1$ for the omega network:

$$\left(\left\langle x,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle, \left\langle x+1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d \right\rangle\right) \in E_{\Omega}$$

for $0 \leq d < m$.

Assume that the mapping for cells in stages $\leq x$ has been found. (See below.)

Then in the recursive omega consider $f(\langle x,\, i\rangle)$, cell i's image.

$$f\left(\left\langle x,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle\right) = \left\langle x,\, i_{(x-1)} \ldots i_{(0)} i_{(n-2)} \ldots i_{(x)} \right\rangle$$

The link from this cell to stage $x+1$:

$$\left(\left\langle x,\, i_{(x-1)} \ldots i_{(0)} i_{(n-2)} i_{(n-3)} \ldots i_{(x)} \right\rangle, \left\langle x+1,\, i_{(x-1)} \ldots i_{(0)} d\, i_{(n-3)} \ldots i_{(x)} \right\rangle\right) \in E_{\mathrm{R}\Omega}$$

for $0 \leq d < m$.

To show $\Omega \stackrel{\mathrm{TE}}{=} \mathrm{R}\Omega$ we must find maps between these stage-$x+1$ cells:

$$\begin{aligned} f(\langle x+1,\, i\rangle) &= \\ f\left(\left\langle x+1,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle\right) &= \left\langle x+1,\, i_{(x-1)} \ldots i_{(0)} i_{(n-2)} \ldots i_{(x)} \right\rangle \\ &= \left\langle x+1,\, \sigma_{m^{n-x-1}, m^{x}}(i) \right\rangle \end{aligned}$$

Using the function:

$$f\left(\left\langle x,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle, \left\langle x+1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d \right\rangle\right) \in f(E_{\Omega})$$

$$\left(\left\langle x,\, i_{(x-1)} \ldots i_{(0)} i_{(n-2)} i_{(n-3)} \ldots i_{(x)} \right\rangle, \left\langle x+1,\, i_{(x-1)} \ldots i_{(0)} d\, i_{(n-3)} \ldots i_{(x)} \right\rangle\right) \in E_{\mathrm{R}\Omega}$$

Wrapup of Proof

Determination of Mapping Function

- Found explicitly for stages 0, 1, and 2.

  It was assumed that a mapping function was found for stages $3 \ldots x < n-1$.

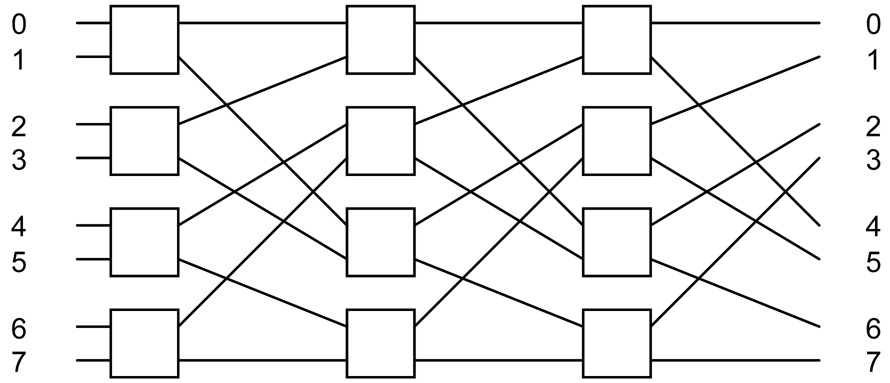
- Using this assumption one could find the mapping for stage $x+1$.

- By induction, mapping can be found for all stages.

Since mapping functions found for all cells, networks are TE.

Proof that Inverse Omega (IΩ) is not TE to Omega

Introducing the inverse omega network:



This is simply the mirror-image of the omega network.

Structure of an n-stage $m \times m$-cell inverse omega network:

- Each stage has m^{n-1} cells.
- Network inputs connect to like-numbered stage-0 cell inputs.
- Stage-x cell outputs connect to stage-$(x+1)$ cell inputs with a m^{n-1}, m shuffle, for $0 \leq x < n-1$.
- Stage-$(n-1)$ cell outputs connect to network outputs with a m^{n-1}, m shuffle.

Routing:

To route request (a, α) use cell output $\alpha_{(x)}$ in stage x.

Inverse omega network LGM:

Let $k = m^{n-1}$.

$$I_{\mathrm{I}\Omega} = \{\, \langle \mathrm{I},\, i \rangle \mid 0 \le i < mk \,\} \qquad O_{\mathrm{I}\Omega} = \{\, \langle \mathrm{O},\, i \rangle \mid 0 \le i < mk \,\}$$
$$V_{\mathrm{I}\Omega} = I_{\mathrm{I}\Omega} \cup O_{\mathrm{I}\Omega} \cup \{\, \langle x,\, i \rangle \mid 0 \le x < n,\ 0 \le i < k \,\}$$

$$\begin{aligned} E_{\mathrm{I}\Omega} = &\left\{ \left(\langle \mathrm{I},\, i \rangle , \left\langle 0,\, \left\lfloor \frac{i}{m} \right\rfloor \right\rangle \right) \mid 0 \le i < mk \right\} \bigcup \\ &\left\{ \left(\langle x,\, i \rangle , \left\langle x+1,\, \left\lfloor \frac{i}{m} \right\rfloor + j\, m^{n-2} \right\rangle \right) \;\middle|\; 0 \le j < m,\ 0 \le i < k,\ 0 \le x < n-1 \right\} \bigcup \\ &\left\{\, \left(\langle n-1,\, i \rangle , \left\langle \mathrm{O},\, i + j\, m^{n-2} \right\rangle \right) \mid\ 0 \le j < m,\ 0 \le i < k \,\right\} \end{aligned}$$

Proof That $\Omega \overset{\mathrm{TE}}{\neq} \mathrm{I}\Omega$.

In omega network: $\langle \mathrm{I},\, 0 \rangle \in I_\Omega$ and $(\langle \mathrm{I},\, 0 \rangle , \langle 0,\, 0 \rangle) \in E_\Omega$.

In inverse omega network: $\langle \mathrm{I},\, 0 \rangle \in I_{\mathrm{I}\Omega}$ and $(\langle \mathrm{I},\, 0 \rangle , \langle 0,\, 0 \rangle) \in E_{\mathrm{I}\Omega}$.

Therefore, $f(\langle 0,\, 0 \rangle) = \langle 0,\, 0 \rangle$.

But $\left(\left\langle \mathrm{I},\, m^{n-1} \right\rangle , \langle 0,\, 0 \rangle \right) \in E_\Omega$ and $(\langle \mathrm{I},\, 1 \rangle , \langle 0,\, 0 \rangle) \in E_{\mathrm{I}\Omega}$.

Since $\left(\left\langle \mathrm{I},\, m^{n-1} \right\rangle , \langle 0,\, 0 \rangle \right) \neq (\langle \mathrm{I},\, 1 \rangle , \langle 0,\, 0 \rangle) \in E_{\mathrm{I}\Omega}$,

$$\Omega \overset{\mathrm{TE}}{\neq} \mathrm{I}\Omega.$$

That's it.

Topological Equivalence

Two networks are *topologically equivalent (OE)* if their LGM representations are identical for some renaming of vertices in V.

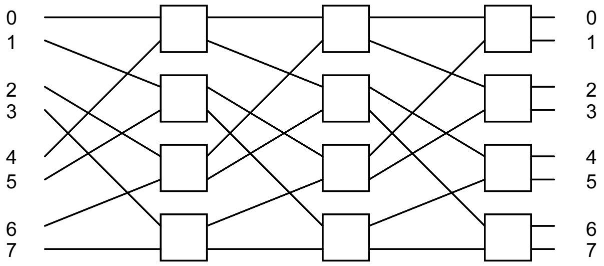
The relational operator $\stackrel{\mathrm{OE}}{=}$ indicates topological equivalence; $A \stackrel{\mathrm{OE}}{=} B$ means A is topologically equivalent to B.

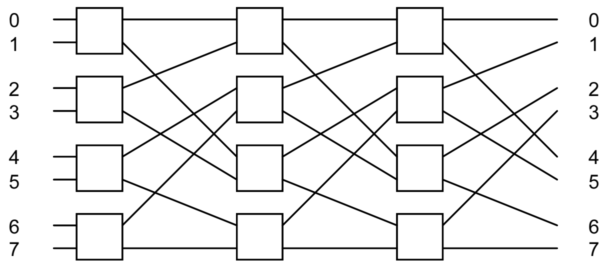
Mathematically,

$$A \stackrel{\mathrm{OE}}{=} B \iff \exists \, (f \mid V_A \to V_B) \ni f(A) = B,$$

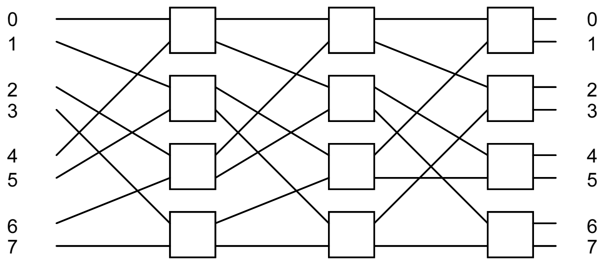
where $A = (I_A, O_A, V_A, E_A)$ and $B = (I_B, O_B, V_B, E_B)$ are network LGMs.

The following two networks are topologically equivalent:





The one below is not OE to either of the two above:



Proof Methods

Let $A = (I_A, O_A, V_A, E_A)$ and $B = (I_B, O_B, V_B, E_B)$ be network LGMs.

Easy cases: The two networks are *not* OE if:

Either $|I_A| \neq |I_B|$, $|O_A| \neq |O_B|$, $|V_A| \neq |V_B|$, or $|E_A| \neq |E_B|$. (The number of inputs, outputs, cells, and links must be identical for OE.)

Hard cases: find the mapping function, f.

For many networks f can be either quickly found or…

…can be shown to not exist.

For others, an exhaustive method must be used.

Differences From TE Proofs

No unique naming of input- and output-stage cells.

Mapping function for input and output terminals must also be found.

Proof That Omega is OE to Inverse Omega

Proof Plan:

- Guess a mapping function for input labels.

  (The first guess presented will be wrong.)

- Compare links in the two networks that connect to the same input.

- Based on this, find mapping for first-stage cells, if possible.

  (If it is not possible we have to choose a different mapping for the input labels. If a different mapping cannot be found then the networks are not OE.)

- Continue to outputs.

  (Backtracking, when necessary.)

(Incorrect) Input Mapping Function

$$f\left(\langle \mathrm{I},\, i\rangle\right) = \left\langle \mathrm{I},\, \sigma_{m,m^{n-1}}(i)\right\rangle$$

Given this mapping function:

First find link to stage 0 in Ω:

$$\left(\left\langle \mathrm{I},\, i_{(n-1)} i_{(n-2)} \ldots i_{(0)}\right\rangle, \left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle\right) \in E_\Omega$$

The analogous link in the inverse omega:

$$\left(\left\langle \mathrm{I},\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} i_{(n-1)}\right\rangle, \left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle\right) \in E_{\mathrm{I}\Omega}$$

The map for the stage-0 cells couldn't be easier:

$$f(\langle 0,\, i\rangle) = \langle 0,\, i\rangle$$

For stage 1:

$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle, \left\langle 1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d\right\rangle\right) \in E_\Omega$$

The analogous link in the inverse omega:

$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)}\right\rangle, \left\langle 1,\, d\, i_{(n-2)} i_{(n-3)} \ldots i_{(1)}\right\rangle\right) \in E_{\mathrm{I}\Omega}$$

The map for the stage-1 cells is impossible since no function could map $\left\langle 1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d\right\rangle$ to $\left\langle 1,\, d\, i_{(n-2)} i_{(n-3)} \ldots i_{(1)}\right\rangle$ (because, for one reason, $i_{(n-2)}$ appears in only one).

(Correct) Input Mapping Function

Input mapping will convert input number using the n-digit, radix-m, *bit reverse function,* $\rho \mid \langle m^n \rangle \to \langle m^n \rangle$:

$$\rho(i_{(n-1)} i_{(n-2)} \ldots i_{(0)}) = i_{(0)} i_{(1)} \ldots i_{(n-1)}$$

In words: digits in radix-m representation are reversed.

Input mapping function:

$$f\left(\langle \mathrm{I},\, i \rangle\right) = \langle \mathrm{I},\, \rho(i) \rangle$$

Given this mapping function:

First find link to stage 0 in Ω:

$$\left(\left\langle \mathrm{I},\, i_{(n-1)} i_{(n-2)} \ldots i_{(0)} \right\rangle, \left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle\right) \in E_\Omega$$

The analogous link in the inverse omega:

$$\left(\left\langle \mathrm{I},\, i_{(0)} i_{(1)} \ldots i_{(n-2)} i_{(n-1)} \right\rangle, \left\langle 0,\, i_{(0)} i_{(1)} \ldots i_{(n-3)} i_{(n-2)} \right\rangle\right) \in E_{\mathrm{I}\Omega}$$

The map for the stage-0 cells is familiar:

$$f(\langle 0,\, i \rangle) = \langle 0,\, \rho(i) \rangle$$

For stage 1:

$$\left(\left\langle 0,\, i_{(n-2)} i_{(n-3)} \ldots i_{(0)} \right\rangle, \left\langle 1,\, i_{(n-3)} i_{(n-4)} \ldots i_{(0)} d \right\rangle\right) \in E_{\Omega}$$

The analogous link in the inverse omega:

$$\left(\left\langle 0,\, i_{(0)} i_{(1)} \ldots i_{(n-2)} \right\rangle, \left\langle 1,\, d\, i_{(0)} i_{(1)} \ldots i_{(n-3)} \right\rangle\right) \in E_{\mathrm{I}\Omega}$$

The map for the stage-1 cells is also familiar:

$$f(\langle 1,\, i\rangle) = \langle 1,\, \rho(i)\rangle$$

It can easily be shown that all nodes are mapped using ρ.

Therefore, $\Omega \overset{\mathrm{OE}}{=} \mathrm{I}\Omega$.

Functional Equivalence

Two networks are *functionally equivalent (FE)* if the sets of connection assignments each can satisfy are identical.

Proof of functional equivalence:

Find a set of connection assignments that each network can satisfy.

Show that the two sets are identical.

This is tedious using LGMs, CGMs, and similar graph notation.

It is much easier to do using sets-of-permutation models, to be covered soon.