Admissibility Function

Function used to test if permutation is in $\Omega_{m,n}$.

Theorem: $P \in \Omega_{m,n}$ iff

there exists functions $f_x \mid \langle m^{n-1} \rangle \rightarrow \Sigma_m$

such that for all $(a, \alpha) \in P$ and $x \in \langle n \rangle$

$$\alpha_{(n-x-1)} = a_{(n-x-1)} f_x \left(a_{(n-x-2)} a_{(n-x-3)} \cdots a_{(0)} \alpha_{(n-1)} \alpha_{(n-2)} \cdots \alpha_{(n-x)} \right)$$

In words:

x is the stage number.

$a_{(n-x-2)} a_{(n-x-3)} \cdots a_{(0)} \alpha_{(n-1)} \alpha_{(n-2)} \cdots \alpha_{(n-x)}$ is the cell number.

f_x is how the cell should be set.

So, P is admissible...

...if there exist cell settings, f_x, such that...

...for each request $(a, \alpha) \in P$...

...and each stage x...

...the cell output on path from a to α is used:

$$\alpha_{(n-x-1)} = a_{(n-x-1)} f_x \left(a_{(n-x-2)} a_{(n-x-3)} \cdots a_{(0)} \alpha_{(n-1)} \alpha_{(n-2)} \cdots \alpha_{(n-x)} \right).$$

Put yet another way:

A permutation is admissible iff a cell's setting can be determined using only its stage number and dual-path descriptor.

Lower- and Upper-Triangular Permutations

Upper- and lower-triangular permutations defined using function.

A permutation λ is (m, n) *lower triangular*...

...iff there exist f_x for $x \in \langle n \rangle$...

...such that for all $(a, \alpha) \in \lambda$,

$$\alpha_{(n-x-1)} = a_{(n-x-1)} f_x \left(a_{(n-x-2)} a_{(n-x-3)} \cdots a_{(0)} \right)$$

where $f_x()$ is a constant.

In such permutations, cell state determined by reverse-path descriptor.

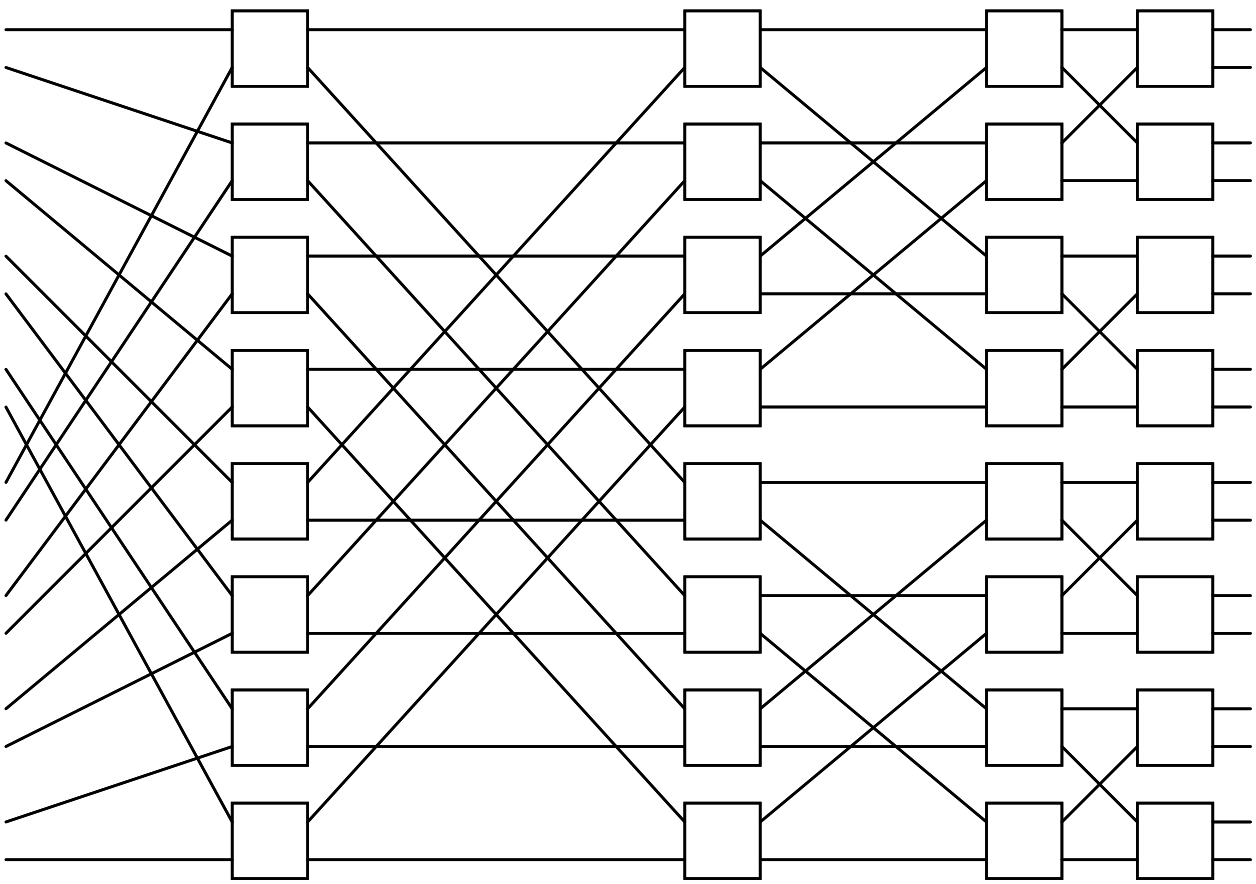
$\Lambda_{m,n}$ will denote set of all (m, n) lower triangular permutations.

Almost by definition, $\Lambda_{m,n} \subseteq \Omega_{m,n}$.

Recursive Construction

Set $\Lambda_{2,n}$ constructed from $\Lambda_{2,n-1}$.

$$\Lambda_{2,n} = \{\sigma_{2,2^{n-1}} E \sigma_{2^{n-1},2} \lambda S_{1,2^{n-1}} \lambda S_{1,2^{n-1}} \mid E \in \mathcal{E}_{2,n}, \lambda \in \Lambda_{2,n-1}\}.$$



A permutation v is (m,n) *upper triangular...*

...iff there exist f_x for $x \in \langle n \rangle$...

...such that for all $(a,\alpha) \in v$,

$$\alpha_{(n-x-1)} = a_{(n-x-1)} f_x \left(\alpha_{(n-1)} \alpha_{(n-2)} \cdots \alpha_{(n-x)} \right)$$

...where $f_x()$ is a constant.

In such permutations, cell state determined by forward-path descriptor.

$\Upsilon_{m,n}$ will denote set of all (m,n) upper-triangular permutations.

Almost by definition, $\Upsilon_{m,n} \subseteq \Omega_{m,n}$.

Groups

Sets of permutations are sometimes *groups* (under composition).

Groups have useful properties and are well studied.

Therefore, it's useful to identify permutation sets as groups.

First, group definition:

A set G and a binary operation $*$ form a *group* if

- $*$ is associative;
- there exists an element e, called the identity element, such that for all $a \in G$ the following hold $e * a = a * e = a$;
- for all $a \in G$ there exists a^{-1}, called the *inverse*, such that $aa^{-1} = a^{-1}a = e$.

Group Property:

Closed under composition: if $a \in G$ and $b \in G$ then $a * b \in G$.

For permutations:

Operation is composition.

The identity is I. (For example, $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.)

The inverse corresponds to the inverse of a permutation.

Set Σ_N is a group.

Set Ω_N is not a group.

Sets Λ and Υ are groups.

One property of a group is that inverse exists.

That is proven on next page for Υ.

Theorem: if $\upsilon \in \Upsilon$ then $\upsilon^{-1} \in \Upsilon$.

Proof:

Let $\upsilon \in \Upsilon$ and $(a, \alpha) \in \upsilon$. By definition the f_x exist:

$$\alpha_{(n-x-1)} = a_{(n-x-1)} f_x \left(\alpha_{(n-1)} \cdots \alpha_{(n-x)}\right)$$

Consider the inverse, υ^{-1}.

Since $(a, \alpha) \in \upsilon$ then $(\alpha, a) \in \upsilon^{-1}$.

To prove $\upsilon^{-1} \in \Upsilon$ must show f'_x such that

$$a_{(n-x-1)} = \alpha_{(n-x-1)} f'_x(a_{(n-1)} \cdots a_{(n-x)})$$

To do that, find g_x:

$$g_x(a_{(n-1)} a_{(n-2)} \cdots a_{(n-x)}) = \alpha_{(n-1)} \alpha_{(n-2)} \cdots \alpha_{(n-x)}$$

Start at first stage, $x = 0$:

$$\alpha_{(n-1)} = a_{(n-1)} f_0\,()$$

Therefore $g_0(a_{(n-1)}) = a_{(n-1)} f_0()$.

In second stage, $x = 1$:

$$\alpha_{(n-2)} = a_{(n-2)} f_1 \left(\alpha_{(n-1)}\right)$$

Therefore:

$$\begin{aligned} g_1(a_{(n-1)} a_{(n-2)}) =& m \times \alpha_{(n-1)} + \left[a_{(n-2)} f_1 \left(\alpha_{(n-1)}\right)\right] \\ =& m \times g_0(a_{(n-1)}) + \left[a_{(n-2)} f_1 \left(g_0(a_{(n-1)})\right)\right] \end{aligned}$$

Assuming g'_x exists for $x' < x$:

$$\begin{aligned} & g_x(a_{(n-1)}a_{(n-2)}\cdots a_{(n-x)}) \\ & \quad = m \times \left[\alpha_{(n-1)}\cdots\alpha_{(n-x)}\right] + \left[a_{(n-x-1)}f_x\left(\alpha_{(n-1)}\cdots\alpha_{(n-x)}\right)\right] \\ & \quad = m\times\left[g_{x-1}(a_{(n-1)}\cdots a_{(n-x)})\right]+\left[a_{(n-x-1)}f_x\left(g_{x-1}(a_{(n-1)}\cdots a_{(n-x)})\right)\right] \end{aligned}$$

And so by induction, g_x exists for all $x \in \langle n \rangle$.

Now, use g_x to find f'_x.

We know:

$$\alpha_{(n-x-1)} = a_{(n-x-1)}f_x\left(\alpha_{(n-1)}\alpha_{(n-2)}\cdots\alpha_{(n-x)}\right)$$

Solving for $a_{(n-x-1)}$:

$$a_{(n-x-1)} = \alpha_{(n-x-1)}\left(f_x\left(\alpha_{(n-1)}\alpha_{(n-2)}\cdots\alpha_{(n-x)}\right)\right)^{-1}$$

Note $\left(f_x\left(\alpha_{(n-1)}\alpha_{(n-2)}\cdots\alpha_{(n-x)}\right)\right)^{-1}$ is inverse of the value, not the function.

Replacing α's:

$$a_{(n-x-1)} = \alpha_{(n-x-1)}\left(f_x\left(g_x(a_{(n-1)}a_{(n-2)}\cdots a_{(n-x)})\right)\right)^{-1}$$

Thus $f'_x(a_{(n-1)}a_{(n-2)}\cdots a_{(n-x)}) = \left(f_x\left(g_x(a_{(n-1)}\cdots a_{(n-x)})\right)\right)^{-1}$.

Since f'_x can be found for any $v \in \Upsilon\ldots$

$\ldots$indicating that $v^{-1} \in \Upsilon,\ldots$

$\ldots$the inverse exists for any element of Υ.

Invariance Properties of Upper- and Lower-Triangular Permutations

The set of lower-triangular permutations form *right-invariants* of Ω.

That is: $P\Omega = \Omega \iff P \in \Lambda$.

Similarly.

The set of upper-triangular permutations form *left-invariants* of Ω.

That is: $\Omega P = \Omega \iff P \in \Upsilon$.